Repeated Contracting without Commitment

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Abstract

I study a dynamic model of monopoly sales in which a monopolist without commitment power interacts with a consumer whose valuation is private. I characterize the equilibrium of this game and show how the seller's strategy varies with initial beliefs. I find that the seller's payoffs from spot contracting can be higher than under commitment with renegotiation, and that random delivery contracts can improve payoffs beyond posted prices.

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1 Introduction

This paper provides a solution to a classic problem in economics: how a monopolist sells a perishable good to a buyer with private valuation when the monopolist cannot commit to future prices. It fully characterizes the equilibrium which is best from the point of view of the seller and shows that several results from the literature are incorrect.

In the model I study, a monopolist seller (she) of a perishable good interacts with a buyer (he) whose private valuation can be either high or low. This interaction is finitely repeated. The seller offers a price in each period and the buyer chooses whether or not to purchase, leading the seller to update her beliefs. I contrast this spot contracting setting, in which the seller cannot commit to price offers in the future, to the full commitment and commitment with renegotiation settings whose solutions are well known. The spot contracting model is equivalent to the "rental model under non-commitment" that was presented in Hart and Tirole (1988). Limiting commitment has the effect of delaying the seller's learning. Identifying a low valuation buyer leads the seller to charge him a low price in all periods, reducing the surplus which can be extracted from high valuation buyers. Spot contracting introduces an additional reverse incentive compatibility constraint in which low types have the incentive to purchase at a low price and then never interact with the seller again. The optimal equilibrium for the seller takes both of these effects into account to determine the optimal sequence of prices to offer.

In the equilibrium which is optimal from the point of view of the seller, there are three outcomes that can occur in a given period. When the buyer is unlikely to have high valuation, the seller pools both types of buyers and charges a low price. When the buyer is very likely to have a high valuation, the seller charges a high price at which a high valuation buyer randomizes between purchasing or not. Finally, for intermediate beliefs the seller sometimes sets a low price at which a high valuation buyer always purchases, but at which a low valuation buyer randomizes the decision to purchase.

The fact that there are beliefs for which high valuation buyers purchase and low valuation buyers mix contradicts results from Hart and Tirole (1988), which claimed that low valuation buyers never mix. Inducing low valuation buyers to mix can increase profits for the seller because it can lead to higher posteriors (which are associated with higher continuation profits) at a relatively small cost in the current period.

Despite being unable to commit to as much, there are some cases in which a spot contracting monopolist makes strictly higher profits than a monopolist who commits with renegotiation. This contradicts Proposition 6 of Hart and Tirole (1988) which claims that in this setting spot contracting is always worse. Payoffs in both settings are lower than under full commitment due to the seller's opportunistic behavior in later periods. In the cases in which spot contracting gives higher payoffs than commitment with renegotiation, the extra constraints actually restrict this opportunistic behavior in future periods, thus allowing the monopolist to make higher payoffs from the point of view of the first period.

I also show that restricting a seller to posted prices can lower the seller's profits under spot contracting. In the cases in which a low valuation buyer randomizes between purchasing and not purchasing, a high valuation buyer strictly prefers to purchase. Thus, the seller can improve payoffs by allowing low valuation buyers to randomize between purchasing and a random delivery contract. This increases the seller's profits without violating the high valuation buyer's incentive compatibility constraints.

Early work in contract theory showed that the contract a monopolist will implement if she has full commitment power cannot be implemented through spot contracts and will not necessarily be renegotiation proof (Baron & Besanko, 1984). Because of this, a large literature has grown around studying dynamic mechanism design in settings where the principal does not have full commitment power. The model this paper focuses on features a monopolist with constant marginal cost and a buyer with unit demand. Other models studied by Laffont and Tirole (1987, 1988, 1990) have an increasing marginal cost structure and unbounded demand, but equilibria are complex and difficult to characterize even for two period, two type models.

The specific spot contracting model which is studied in this paper was first solved by Schmidt (1993) in an equivalent setting for the case in which both the buyer's and seller's discount rates are equal to one. This assumption on the discount rates implies that the reverse incentive compatibility constraint is *always* binding. This simplifies the problem and implies that all of the seller's learning occurs at the end of the relationship. Furthermore, the assumption on discount rates implies that there are never cases in which low types mix and that the payoffs under spot contracting are always equal to those under commitment with renegotiation. Neither of these results hold when allowing for discount rates strictly less than one. Devanur, Peres, and Sivan (2019) also study the finitely repeated version of this game with no discounting, but focus on continuous distributions of types and compare the spot contracting outcome to a situation in which the seller can commit to not raising prices.

Beccuti and Möller (2018) also study the same interaction between a seller with limited commitment and a buyer with private information, but make a few key assumptions which lead to a different characterization. First, they assume that the buyer has a discount factor which is less than one half (compared to $\delta > \frac{1}{2}$ assumed here). In this case, the reverse incentive compatibility constraint which drives many of the features of the spot contracting equilibrium in this paper *never* binds. Furthermore, they assume that the seller's discount factor is higher than the buyer's and that the seller's initial beliefs are relatively low ($\mu < \frac{\underline{b}}{\bar{b}}$ in the terminology of Section 2). The differences in discount rates lead the seller to extract surplus in future periods by using random delivery contracts in the present period. The incentive to screen in this way is not present when the buyer discounts at the same rate as the seller, which is what is assumed in this paper.

Gerardi and Maestri (in press) study a related model of employment contracting with limited commitment. The interaction is infinitely repeated, and payoffs are nonlinear in the allocation. A key assumption which differs from the model here is that if the employee rejects contract offers in a particular period, they can never again interact with the employer. This in combination with the different payoff structures and infinite repetition leads to either full pooling or immediate full separation when parties are patient enough. In the model studied here, if a consumer who chose not to purchase in a particular period could never purchase again, the seller would receive their full commitment payoffs even in the spot contracting game.

This paper's results show that giving a seller more (but not full) commitment power can lower the seller's payoffs. This relates to previous work which shows how improving contracting within or across periods can lead to worse outcomes (Baker, Gibbons, & Murphy, 1994; Schmidt & Schnitzer, 1995; Kovrijnykh, 2013; Breig, 2019). However, the mechanisms in this previous work revolve around how improving commitment increases the payoffs of the punishment equilibrium in an infinitely repeated game, making deviations from an implicit contract more tempting. Implicit contracts play no role in this paper because the interaction is finitely repeated.

The remainder of the paper proceeds as follows: Section 2 of this paper presents the underlying economic framework. Section 3 characterizes the seller's optimal equilibrium of the spot contracting setting, compares this equilibrium to the solution under other forms of commitment power, and shows that the seller can sometimes improve profits by using a random delivery contract. Section 4 is a conclusion. The proofs that are not in the main body of the text are in Appendix A.

2 Model

A seller (she) and buyer (he) interact for $T < \infty$ periods. In each period, the seller can produce a perishable consumption good at a normalized cost of 0.[1] The buyer has unit demand for the good in each period. The buyer's value of consumption is $b \in \{\underline{b}, \bar{b}\}$, where $0 < \underline{b} < \bar{b}$. This valuation is constant, known to the buyer, and unknown to the seller. The probability that a buyer is of the high type is $P(b = \bar{b}) = \mu$, and to make the problem non-trivial, we will always assume that $\mu\bar{b} > \underline{b}$. Both the buyer and the seller have discount factor δ which is strictly between $\frac{1}{2}$ and 1.[2]

The seller's strategy space in any given period depends on the commitment structure they face. This paper will compare the equilibrium of the *spot contracting* game to equilibria of the *full commitment* and *commitment with renegotiation* games. In the spot contracting game, the seller can post a price which the buyer may pay for consumption in the current period.[3] The seller cannot make any commitments about future periods. With full commitment, the contracts can specify whether the good will be consumed in each period and the price of this sequence of consumption choices. Under commitment with renegotiation, contracts can specify whether the good will be consumed and price levels in all future periods, but a contract can be renegotiated in any future period.

The buyer's strategy space in each period depends on the actions taken by the seller. If the seller did not make an offer in a given period, then the buyer has no choice to make and any previously agreed upon contract is implemented. If, on the other hand, at least one contract is offered in a given period, the buyer can select among the contracts that are offered to him and the continuation contract.

The seller's payoffs are the expected discounted sum of payments which are implemented in equilibrium. The buyer's payoffs are the expected discounted sum of consumption utility minus transfers. I make one additional assumption which restricts the considered set of discount rates.

Assumption 1 *There are no k_1, k_2,…, k_{T-1}, $k_i \in \{-1, 0, 1\}$, such that $k_1\delta + k_2\delta^2 + \cdots + k_{T-1}\delta^{T-1} = 1$.*

Assumption 1 rules out a finite subset of discount rates. The assumption is used in the

[1] The fact that the good is perishable makes the problem different from the literature on bargaining over durable good pricing (Skreta, 2006, 2015; Doval & Skreta, 2020). In particular, the reverse incentive compatibility constraint discussed below can only become binding if the buyer and seller continue interacting after the buyer's type is revealed.

[2] For discount rates that are less than or equal to $\frac{1}{2}$, the reverse incentive compatibility constraint never binds, so the spot contracting equilibrium is the same as the commitment with renegotiation equilibrium.

[3] As will be discussed in Section 3.2, restricting the seller to posted prices is not without loss of generality.

proof of Lemma 1, which characterizes the possible outcomes in each period. Discussion of this assumption will be deferred to the discussion of Lemma 1 in Section 3.

I will study the Perfect Bayesian Equilibria of these game and focus on the equilibria which are optimal from the point of view of the seller. The equilibrium concept requires that given beliefs about the other player's strategy and the buyer's type, both the buyer and the seller are maximizing their expected payoffs. The seller's beliefs about the buyer's type is required to satisfy Bayes' rule whenever possible and only update when the buyer takes an action.[4]

In Section 3 I document properties of the equilibrium of the spot contracting game for particular parameterizations. Example 1 assumes that $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. Example 2 assumes that $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. I specify the strategies which are part of the seller-optimal equilibria for these examples in Appendices B.1 and B.2, respectively.

3 Results

The equilibrium of the full commitment game is well known.

Proposition 1 *In the equilibrium of the full commitment game, the high type purchases the item and pays $\bar{b}$ in each period, and the low type does not consume and pays nothing.*

When priors about the likelihood of the high type are high enough, the optimal full commitment mechanism has the property that the monopolist sells only to the high type and extracts full surplus from him. Notice that the optimal mechanism gives the same allocation to the buyer in each period. This result is also found in Baron and Besanko (1984), and is a function of the fact that the buyer's type is constant. If neither the buyer nor the seller knew the buyer's type in future periods, the seller could extract full surplus from the buyer by simply offering to give the efficient allocation and charging the expected surplus.[5]

Another feature of the optimal mechanism is its allocative inefficiency: the seller's marginal cost is zero, and if nothing else changed, increasing the low type's allocation and the price paid would make the seller strictly better off without affecting the low type's individual rationality constraint. However, this inefficiency allows the seller to extract higher surplus from the high type.

[4] If the seller ever observes an event which should happen with probability zero in equilibrium, I assume that she updates her beliefs to one. This leads to continuation payoffs of zero for both types of buyers.

[5] When an agent's type is serially independent, optimal contracts are generally renegotiation proof. Renegotiation proofness can be maintained even with some serial correlation, but as the likelihood that an agent's type remains the same increases, a lack of commitment eventually induces a distortion (Battaglini, 2005, 2007).

A further consequence of this inefficiency is that the optimal full commitment mechanism is not renegotiation proof. If the seller were attempting to implement this full commitment mechanism in a setting where she was allowed to renegotiate contract offers, she would not be able to. Entering the second period, the monopolist has full information about the buyer's type; if the buyer revealed himself to have a low valuation by not purchasing in the first period, the seller would prefer to offer him a new contract with an efficient quantity and charge $\underline{b}$. This is would be acceptable to the low type because he is not receiving any surplus in either case. Because the high type knows that the low type's contract will be renegotiated in the second period, he will not accept the contract in the first period, and the seller cannot implement this allocation.

In a sense, when the contract can be renegotiated, attempting to implement the full commitment allocation causes the seller to learn too fast. She cannot help but take advantage of the information she knows in the second period as a result of trying to implement the full commitment allocation. Hart and Tirole (1988) show that if priors are high enough, the optimal mechanism does not learn the buyer's types immediately as in the full commitment case, but instead spreads the learning out over a number of periods so that the seller can extract higher surplus from the high types.

Proposition 2 *The equilibrium path of the commitment and renegotiation game is generically unique and takes the following form: there exists a sequence of numbers* $0 = \bar{\mu}_0 < \bar{\mu}_1 < \cdots < \bar{\mu}_T < 1$ *such that*

(i) If current posterior beliefs μ_t *at date* t *belong to the interval* $[\bar{\mu}_i, \bar{\mu}_{i+1})$ *for* $i \leq T - t + 1$*, the seller will sell only to high types for* i *more periods including the current one. Posterior beliefs are* $\bar{\mu}_{i-1}$ *at* $t + 1$*,* $\bar{\mu}_{i-2}$ *at* $t + 2$ *and so on. The discounted sum of prices charged in one of these periods is such that the high typed buyer is indifferent between purchasing and waiting for the low type's contract.*

(ii) If current beliefs are such that $\mu_t \geq \bar{\mu}_{T-t+1}$*, only high types purchase in every period, and the discounted sum of prices charged is such that the high type is indifferent between his allocation and not purchasing.*

Proof Hart and Tirole (1988). □

Proposition 2 states that when T is high enough, for a fixed number of periods independent of T the seller makes sales only to a buyer with a high valuation, after which she sells to all buyers. The discounted sum of prices charged to a high types makes him indifferent between his allocation and mimicking a low type. The likelihood a high valuation buyer purchases in a given period is chosen to optimally trade off between increasing the likelihood of sales to a high type early in the relationship and and putting off learning about the buyer's type so that the seller can extract more surplus from high valuation buyers.

The equilibrium described in Proposition 2 requires that the buyer and seller be able to at least partially commit for the future. If instead the seller were only able to commit to spot contracts, there are combinations of parameters for which the equilibrium outcome described in Proposition 2 would not be incentive compatible. Without any commitment power between periods, the seller will always charge the buyer $\bar{b}$ after learning that a buyer has a high valuation. Thus, to induce such a buyer to reveal their type the seller must charge the buyer a price which leaves the buyer surplus equal to the discounted sum of the low valuation buyer's future consumption (this is exactly the price which will make him indifferent between accepting his offer and waiting for another offer).

Suppose that a monopolist were attempting to implement this equilibrium with only spot contracts. If some period t is the final period in which she is selling only to high types, then p_t must satisfy

$$\bar{b} - p_t \geq (\bar{b} - \underline{b}) \sum_{k=1}^{T-t} \delta^k$$

so that the surplus the high type receives in period t (the left hand side) is greater than what he would receive if he didn't purchase, instead consuming in each future period at a price of $\underline{b}$. However, when $\delta > \frac{1}{2}$ and T is large enough, this requires p_t to be strictly below $\underline{b}$. Notice that since this contract leaves zero surplus for the low types, when they are offered a price of consumption below $\underline{b}$, the low types would also choose to consume in that period (preventing the monopolist from identifying and charging high prices to the high types in the future).

Thus, the spot contracting setting introduces an additional difficulty to contracting: the ability of the buyer to exit the contract after a low price offer, or "take-the-money-and-run" (Laffont & Tirole, 1987). Identifying all of the high types early in the relationship requires offering them a contract which the low types would also prefer to take. To prevent this reverse incentive compatibility constraint from being violated, the seller might have to wait until later in the relationship to learn the buyers' types. Otherwise, she would have to charge too low of a price.

The first step towards characterizing the equilibrium of the spot contracting game involves ruling out some of the potential outcomes in a given period by showing that they are either infeasible or suboptimal. Lemma 1 guarantees that regardless of the continuation equilibrium, the seller need only compare the optimal payoffs from three types of outcomes.

Lemma 1 *In any any period of any equilibrium of the spot contracting game, the seller either*

1. *fully pools with both types buying,*
2. *separates such that high types buy and low types mix between buying and not buying, or*
3. *separates such that low types do not buy and high types buy or mix between buying and not buying.*

Lemma 1 is the point at which I use Assumption 1. In particular, the assumption is used to rule out any outcomes which involve *both* high valuation and low valuation buyers randomizing whether or not they buy in a given period. Having buyers randomize in this way may be optimal from the point of view of the seller because the associated posteriors may lead to higher continuation values without violating incentive constraints. Assumption 1 guarantees that this randomization will violate the incentive constraints of high valuation buyers. In particular, the only price which may induce low valuation buyers to randomize is $\underline{b}$. Thus, by purchasing high valuation buyers will receive $\bar{b} - \underline{b}$ plus a continuation value, while by not purchasing they receive only their continuation value. However, without discount rates which satisfy the equality from Assumption 1, there are no continuation equilibria which will make high valuation buyers indifferent about purchasing in a given period.

This lemma also uses the fact that I restrict the seller to offering posted prices. If the seller had a broader set of options available to her (for instance, offering contracts with random delivery), the optimal set of outcomes in a period could be quite different. In fact, random delivery can relax incentive constraints or strictly improve payoffs as will be discussed in Section 3.2.

With the simplification given by Lemma 1, I can characterize the equilibrium of the spot contracting game.

Proposition 3 *In the optimal equilibrium of the spot contracting game there exist numbers* $\underline{\mu}_t$ *and* $\hat{\mu}_t$*,* $0 < \underline{\mu}_t \leq \hat{\mu}_t < 1$ *such that*

- *for* $\mu \geq \hat{\mu}_t$ *the low types do not buy and the high types either mix or buy with probability one,*
- *for* $\mu \in (\underline{\mu}_t, \hat{\mu}_t)$ *either both types buy or the high types buy and the low types mix,*
- *for* $\mu \leq \underline{\mu}_t$ *both types buy.*

The proof uses backwards induction starting from period $T-1$. It shows that if continuation values starting from period $t+1$ (as a function of beliefs in period $t+1$) take a particular form, then the optimal outcome in that period takes the form described in the proposition and continuation values starting from period t take the same form. The hypothesized form of continuation values for the seller is increasing, convex, and piecewise linear. While the pooling outcome is straightforward, the other two potential outcomes merit a discussion.[6]

For high enough beliefs in any period t of the spot contracting game, only high types buy with positive probability. If the seller observes the buyer purchasing the item, she updates her beliefs to 1, while she updates her beliefs downward otherwise. Given that upon purchasing the seller knows the type of the high valuation buyer, she charges $\bar{b}$ in all future periods. Thus, the buyer must receive surplus in period t exactly equal to the continuation value she would receive from choosing not to buy, which pins down the price. The tradeoffs that the seller faces between higher prices and higher continuation values are very similar to those found in the commitment with renegotiation case discussed by Hart and Tirole (1988).

When current beliefs are low it may be impossible to have high types mix and low types not buy. This is because upon observing the buyer not purchase, beliefs must fall even further which leads to higher continuation values for the high valuation buyers. If these continuation values are high enough, then the price the seller would have to charge in the current period to incentivize high types to buy would need to be lower than $\underline{b}$, violating the reverse incentive compatibility constraint.

For some intermediate levels of beliefs, it is optimal for the seller to have low valuation buyers mix between buying and not buying while all high valuation buyers purchase the item with probability one. For low valuation buyers to be willing to mix, this implies that the price being charged is $\underline{b}$. Thus, this equilibrium outcome does not benefit the seller due to earnings in the current period: she actually receives lower payoffs in the current period than she would receive if she charged $\underline{b}$ and all buyers purchased the item. Instead, the benefits come from mixing over continuation payoffs, which are increasing and convex in beliefs.

When current beliefs are high, it may in turn be impossible to implement the outcome in which high types buy and low types mix. High types prefer to buy in this case because the surplus from consuming in the current period at price $\underline{b}$ is higher than the continuation value they would receive if beliefs in the next period are zero. Upon observing a purchase in the current period, posterior beliefs increase leading to lower continuation values for high valuation buyers. If a high valuation buyer's continuation values for all higher beliefs are low enough, then he would have an incentive to deviate.

The optimal payoffs in each period can be calculated by taking the upper envelope of payoffs

[6]Solving for both the "high types mixing" and "low types mixing" equilibria involve finding rates of mixing which maximize expected payoffs for the seller given equilibrium constraints. This could alternatively be framed as optimally choosing *posterior beliefs* as in Doval and Skreta (2018).

from the three possible outcomes. In the proof of Proposition 3, I show that when beliefs are low enough, pooling dominates, while when beliefs are high enough, having high valuation buyers mix dominates. It *can* be the case that for intermediate beliefs, having low types mix maximizes payoffs.

Figure 1 shows the payoffs from each possible outcome when $T = 3$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. This corresponds to the subgame which begins in the *second* period of the spot contracting game in Example 1.[7] In this case, having low types mix is never beneficial to the seller. For beliefs that are low enough (less than $\frac{1}{3}$) the seller pools all buyers at price $\underline{b}$ in the first period. When beliefs are higher than $\frac{1}{3}$, she either partially or fully separates buyers in the first period.

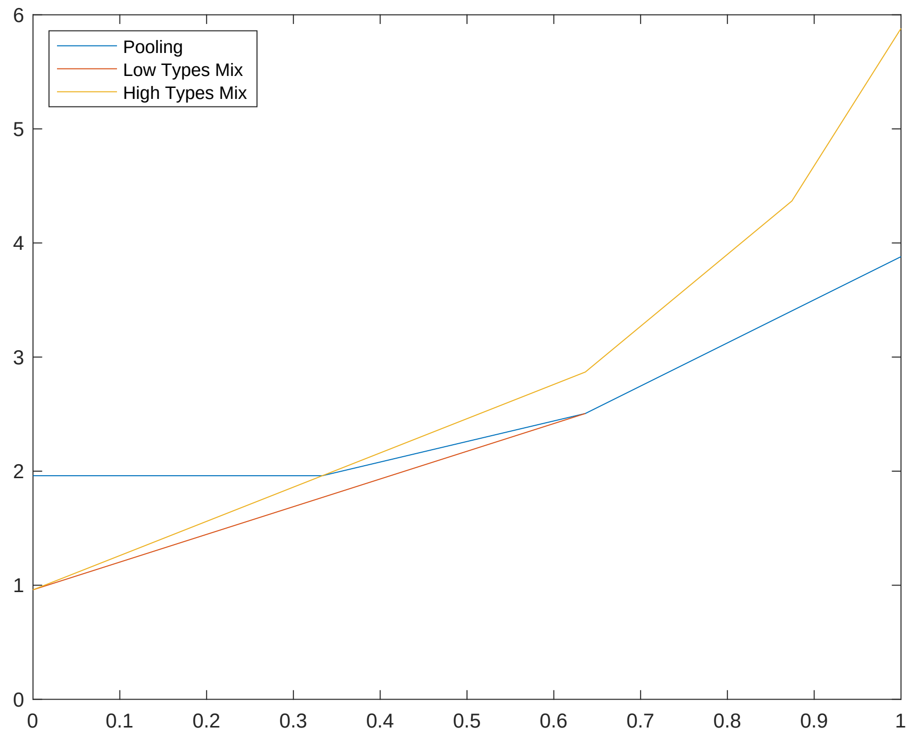

Figure 1: Payoffs for the seller in the first period as a function of μ when $T = 3$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$ for various potential outcomes. Pooling agents is optimal for low beliefs, while having high types mix is optimal for high beliefs.

Figure 2 shows the payoffs from each possible outcome for Example 1, in which $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. It should be noted that the upper envelope of payoffs from Figure 1 form the continuation payoffs to the seller in the first period of a four period game (for instance, the "Pooling" payoffs in Figure 2 are the upper envelope from Figure 1 multiplied by $\delta = 0.6$ and

[7]The equilibrium associated with these parameter values is fully described in Appendix B.1.

added to $\underline{b} = 1$). In this case, having low types mix dominates the other options for intermediate levels of beliefs.

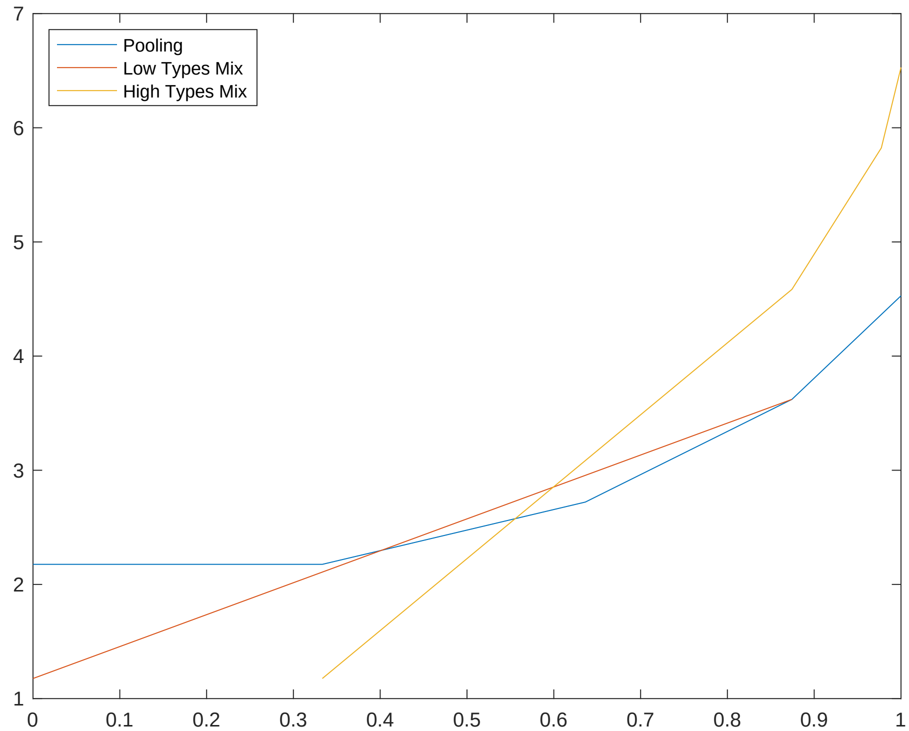

Figure 2: Payoffs for the seller in the first period as a function of μ when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$ for various potential outcomes. Pooling agents is optimal for low beliefs, having low types mix is optimal for intermediate beliefs, and having high types mix is optimal for high beliefs.

The equilibrium path described by Proposition 3 can involve randomization on the part of either buyer, but never involves randomization on the part of the seller. However, the seller does randomize *off* the equilibrium path in a way that induces buyers to mix at the correct rates. A discussion of why this type of randomization is necessary can be found in Gul, Sonnenschein, and Wilson (1986).

The evolution of beliefs was a key part of the description of the equilibrium in Hart and Tirole (1988), and the results from the spot contracting setting can be compared to that equilibria. Beliefs dynamics for the two settings can be found in Figures 3 and 4 for the parameterization given in Example 1. The first point to observe is that these figures are exactly the same starting from period 2. This is due to the fact that with $\delta = 0.6$, $\delta + \delta^2 < 1$ and the reverse incentive compatibility constraint is never binding from period two onward. In this case, the seller implements the same equilibria in both the spot contracting and commitment with renegotiation settings.

The difference between Figures 3 and 4 centers around the evolution of beliefs when $\frac{1}{3} < \mu < \frac{7}{11}$. In the case with renegotiation, the seller would make a price offer which is accepted by all high valuation buyers and no low valuation buyers. Since the seller then has full information about buyers, this offer must offer surplus to the high valuation buyer which is equal to $(\delta+\delta^2+\delta^3)(\bar{b}-\underline{b})$. This cannot be implemented in the spot contracting case because it would involve charging a price below $\underline{b}$. Instead, the seller widens the range of beliefs for which she pools buyers (from all μ less than $\frac{1}{3}$ to all μ less than $\frac{167}{416}$) and the range for which she induces a posterior belief of $\frac{1}{3}$ in the second period (from $\frac{7}{11} \leq \mu < \frac{167}{191}$ to $\frac{3507}{5851} \leq \mu < \frac{167}{191}$). In the remaining undefined range, the seller induces actions which are *never* optimal in the renegotiation case: low valuation buyers randomize while high valuation buyers purchase with probability one.

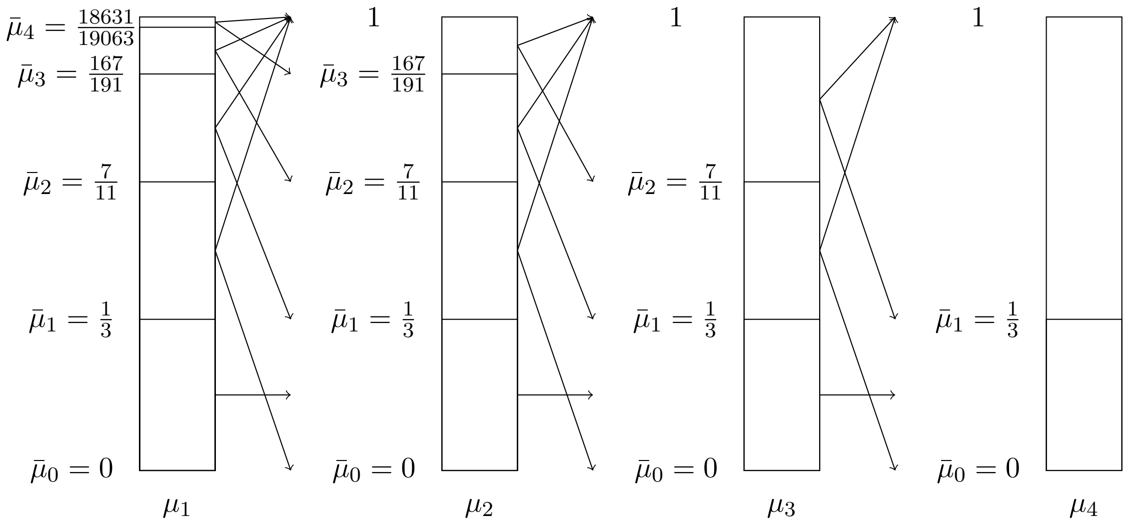

Figure 3: This figure describes the path of beliefs in the renegotiation setting when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. When beliefs are above $\bar{\mu}_1$, the seller offers prices such that high types mix. When the seller observes a purchase, beliefs update to one and otherwise fall to a lower value in the next period. If beliefs are below $\bar{\mu}_1$, the seller always charges $\underline{b}$, both types of buyers purchase with probability one, and the seller does not update.

The existence of an equilibrium in which it is optimal for the seller to have low types mix leads to the first contradiction between the results here and those found in Hart and Tirole (1988).

Observation 1 *Suppose that s satisfies $\delta + \cdots + \delta^{T-s} > 1$. Contrary to Lemma 1 of Hart and Tirole (1988), there can exist a $t \leq s$ such that $\mu_{t+1} < \bar{\mu}_1$ with positive probability.*

Proof Any time the high types buy with probability one and low types mix between buying and

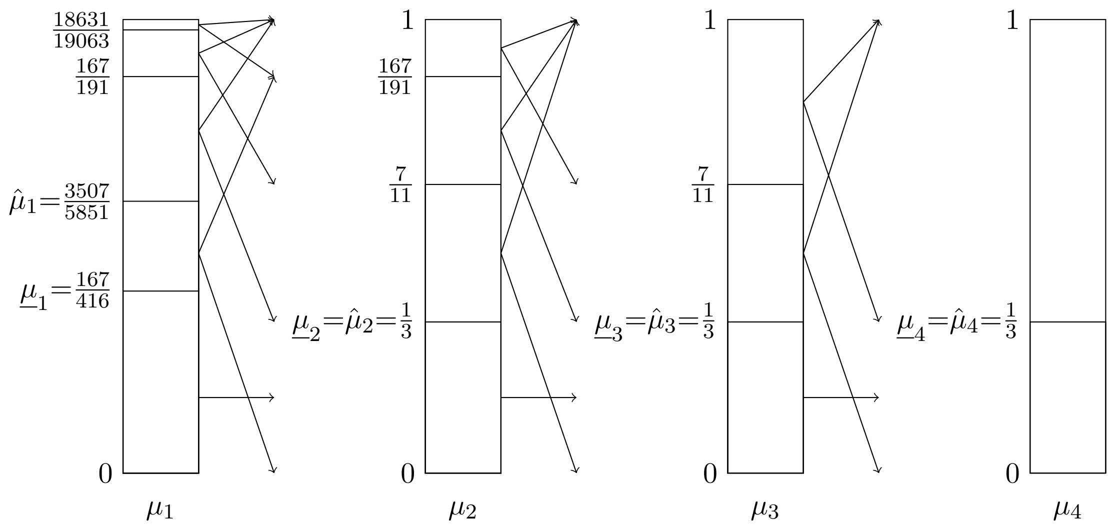

Figure 4: This figure describes the path of beliefs in the spot contracting setting when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. When beliefs are below $\underline{\mu}_t$ the seller pools all buyers and beliefs do not update. When beliefs are above $\hat{\mu}_t$, the low valuation buyer does not purchase and the high valuation buyer purchases with positive probability, implying that beliefs either increase to one or decrease. In the intermediate case in period one, low types are mixing and high types are buying with probability one, so beliefs either increase or fall to zero.

not buying, beliefs fall to 0 when the seller observes the buyer not purchasing. In the equilibrium of Example 1, this outcome occurs in the first period for beliefs μ such that $\frac{167}{416} \leq \mu < \frac{3507}{5851}$. Since in this case $\delta + \delta^2 + \delta^3 > 1$, Lemma 1 of Hart and Tirole (1988) claims that beliefs cannot fall below $\frac{\underline{b}}{\bar{b}}$ before period 3. Thus, the solution to Example 1 provides a counterexample. $\square$

The problem with the proof of Lemma 1 of Hart and Tirole (1988) lies with the claim that low valuation buyers must purchase with probability one in period t when $p_t \leq \underline{b}$ ($r_t \leq \underline{b}$ in their notation). This is true for $p_t < \underline{b}$, but low valuation buyers will be indifferent when $p_t = \underline{b}$. The example shows that it can be valuable to the seller for low valuation buyers to randomize in this case.

The benefit of inducing low valuation buyers to mix is that it allows the seller to have a chance for their beliefs to be high in the next period. Since continuation values are convex in beliefs, inducing randomness in the next period's beliefs is beneficial. However, this comes at the cost of selling only to a proportion of the low valuation buyers in the current period. Furthermore, beliefs in the next period cannot be so high that high valuation buyers would prefer to mimic a low valuation buyer, receiving nothing in the current period and consuming at price $\underline{b}$ in all future periods. The tradeoffs that the seller faces imply that the seller only induces low types to mix when the end of the interaction is at least three periods away.

Corollary 1 *Low types mix in period t only if $T - t \geq 3$.*

Intuitively, when the seller is inducing low types to mix, she is randomizing between two options: receiving nothing in the current period and the continuation payoffs associated with zero beliefs going forward or receiving $\underline{b}$ and the payoffs of higher beliefs going forward. This second option is exactly the payoffs that she receives from pooling both types at those higher beliefs. This can be seen in Figure 2 in that the payoffs from low types mixing are a straight line between payoffs below pooling at current beliefs of zero, and a value exactly equal to one of the kinks of the pooling payoffs at higher beliefs. Noting that these mixing payoffs must be below the pooling payoffs at $\frac{\underline{b}}{\bar{b}}$ (because pooling is the commitment outcome there), one can see that having low types mix can only be profitable if there are at least three kinks in the continuation payoffs, which only occurs when there are at least three periods left.

3.1 Payoffs

The payoff structure as a function of beliefs that arises from spot contracting takes the same form as the payoff structures from full commitment and renegotiation. The payoff function is weakly increasing, piecewise linear, and convex. Furthermore, the payoffs from all three commitment settings are equal to each other when beliefs are equal to one (the seller is sure that all buyers are high types) and when beliefs are below $\frac{\underline{b}}{\bar{b}}$ (when the commitment outcome is to pool all buyers in all periods). Payoffs for the three types of games can be found in Figure 5 for the parameterization given in Example 2.[8]

As is to be expected, the payoffs from full commitment are always weakly higher than the other two settings: with full commitment, the seller could always commit to carry out exactly the same sequence of prices that she would carry out with less commitment, ensuring that she receives the same payoffs. However, despite the fact that commitment with renegotiation in a sense allows for more commitment than spot contracting, payoffs in these two settings cannot consistently be ranked: the payoffs from Example 2 show that for different ranges of prior beliefs, either commitment structure can dominate the other. This fact again contradicts previous results which have been found for this model.

Observation 2 *Contrary to Proposition 6 of Hart and Tirole (1988), payoffs from the commitment*

[8] The equilibrium associated with these parameter values is fully described in Appendix B.2.

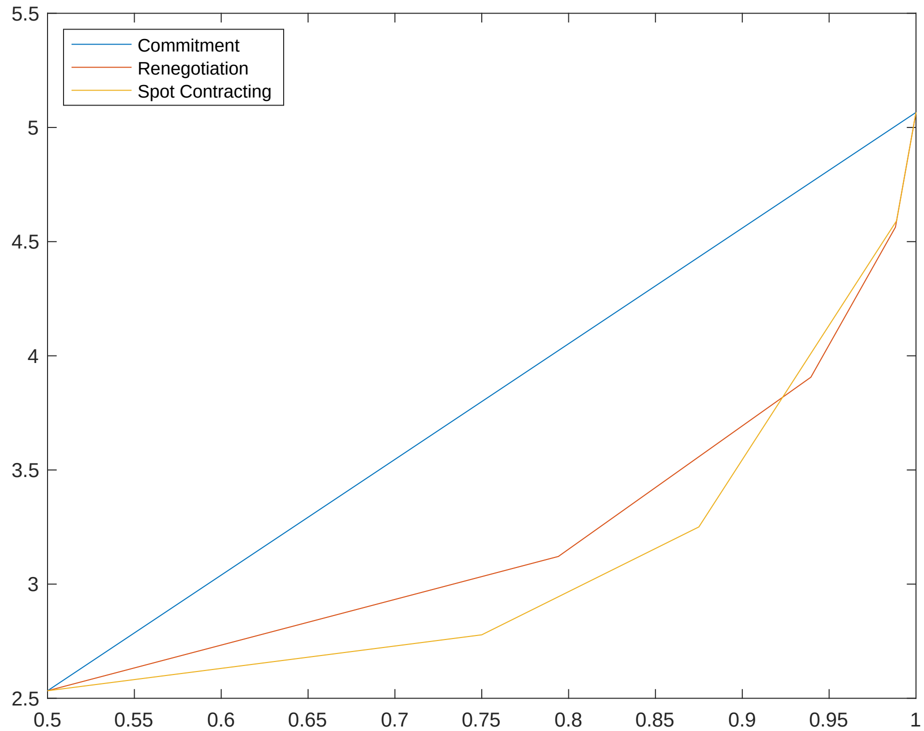

Figure 5: Payoffs for the sellers of various commitment types as a function of μ when $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. Full commitment must always give higher payoffs than other commitment types, but payoff rankings otherwise depend on the specific parameters used.

with renegotiation setting can be strictly below those from the spot contracting setting.

It is instructive to work through exactly when spot contracting has a chance to be better for the seller than commitment with renegotiation. For fixed continuation values, the spot contracting commitment structure simply adds the "take-the-money-and-run" constraint that for an identified high type, information rents must be given in the current period through a price which can be no lower than $\underline{b}$. Since δ is assumed to be strictly less than one, this constraint never binds in period $T-1$, and continuation values from that point forward are the same with both commitment structures. Thus, in period $T-2$, a spot contracting seller is weakly worse off than a seller under renegotiation, because she faces the same continuation values but more constraints.

The earliest period at which spot contracting payoffs can be higher than renegotiation payoffs is $T-3$ (which corresponds to the first period in Example 2). The benefit which can arise for the spot contracting seller in this period comes from the constraint that their future self will face in the next period. The extra "take-the-money-and-run" constraint restricts the opportunistic behavior that the seller would engage in starting from period $T-2$ and allows them to extract more surplus in period $T-3$.

It is important to note that the contradiction of Proposition 6 of Hart and Tirole (1988) does *not* rely on the earlier contradiction pointed out in Observation 1. In fact, for the parameters used in Example 2, inducing low valuation buyers to mix is never beneficial and the possibility of them mixing does not affect payoffs.

The proof of Hart and Tirole's Proposition 6 is completed by backwards induction, with an inductive hypothesis that (in the terminology used here) for any period t, beliefs at that date, and continuation equilibrium in the spot contracting setting, there exists a renegotiation proof outcome which gives the same utilities to both types of buyer and weakly higher payoffs to the seller. They then claim that given that the inductive hypothesis holds for period $t+1$, one can construct a renegotiation proof outcome which dominates any spot contracting outcome. This construction uses the spot outcome from the current period and the renegotiation outcome starting from period $t+1$. While the proof claims that this construction must also be renegotiation proof, in actuality it need not be. In Example 2, when beliefs are between $\frac{3}{4}$ and $\frac{27}{34}$ in the second period it is optimal in the spot contracting game for high valuation buyers to mix such that the posterior is $\frac{1}{2}$ in the third period. However, in the commitment with renegotiation game this is not renegotiation proof because the seller receives higher profits from high valuation buyers purchasing the item with probability one.

3.2 Posted Prices

This paper follows much of the previous literature in assuming that the seller is restricted to posting prices. The important restriction is that in each period a buyer either receives the item or does not; the seller does not have the ability to randomize whether the item will be delivered.

The restrictiveness of this assumption has been discussed in previous work. For instance, posted prices has been shown to be optimal in some cases (Skreta, 2006; Doval & Skreta, 2020), but there are situations in which the seller can improve payoffs by offering a price which leads only to a possibility of the good being delivered (Beccuti & Möller, 2018). In the spot contracting setting described in this paper, I can show that the seller may strictly improve her payoffs at some points by using random delivery of the good.

Proposition 4 *Assuming that the seller posts prices is with loss of generality.*

Proof Take any equilibrium in which high types are buying with probability one and low types are mixing. The strategies of such an equilibrium are given in Appendix B.1. Generically in these cases the low types receive 0 regardless of their choice and the high valuation buyers strictly prefer to buy. In this case, the seller could improve profits by charging price $\varepsilon \underline{b}$ for likelihood ε of receiving the good. For ε small enough, the low valuation buyer can mix at the same rate, no incentive constraints are violated, and payoffs for the seller strictly increase. $\square$

As shown in the proof, the simplest example of when the seller can improve payoffs by not posting prices is when she is inducing low types to randomize, as in Example 1 for for $\frac{167}{416} < \mu < \frac{3507}{5851}$. In this case, the seller offers the item at a price of $\underline{b}$. High valuation buyers purchase the item with probability one and low valuation buyers purchase with probability strictly between zero and one. High valuation buyers receive surplus $(1+\delta^3)(\bar{b}-\underline{b})$ from buying the item, while they would receive a surplus of $(\delta + \delta^2 + \delta^3)(\bar{b} - \underline{b}) < (1 + \delta^3)(\bar{b} - \underline{b})$ if they chose not to purchase in period one. Thus, rather than having low valuation buyers randomize between buying the item at price $\underline{b}$ or not buying the item at all, the seller could offer a contract costing $(1-\delta-\delta^2)\underline{b}$ which gives a chance of $1-\delta-\delta^2$ to receive the item. With this change, the low valuation buyer still earns payoffs of zero from each of his options. The high valuation buyer is made indifferent (although he still selects the higher priced contract with probability one). The seller's expected payoffs increase by $(1-\delta-\delta^2)\underline{b}$ times the likelihood that the buyer would not have purchased the item previously.

The above example shows one simple way that allowing for random delivery can improve payoffs for the seller. However, there may be other more subtle ways in which relaxing the posted prices constraint change the solution to the seller's problem. In particular, one implication of this constraint (in combination with Assumption 1) is that the seller *cannot* implement a "double mixing" equilibrium in which both low and high valuation buyers select into two contracts because it is impossible to promise the necessary continuation values. Without the constraint, it may be optimal to implement such a double mixing equilibrium.

4 Conclusion

In this paper I characterize the seller-optimal equilibrium of a spot contracting game between a monopolist and a consumer with private information. Some of the results contradict previous claims which have been made about the same model. In every period, one of three outcomes occurs: both types are buying, low types are randomizing and high types are buying, or only high types are randomizing. Payoffs in the spot contracting game can be higher than a game with commitment and renegotiation, and the seller can improve their payoffs by not restricting herself to posted prices.

Further work should study under what conditions limited commitment leads to posted prices not being optimal and what form these more general contracts take. This paper shows that simple random delivery contracts can improve profits, but a full mechanism design approach may lead to contracting dynamics which are not seen here.

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A Proofs

Proposition 1 *In the equilibrium of the full commitment game, the high type purchases the item and pays $\bar{b}$ in each period, and the low type does not consume and pays nothing.*

Proof Notice that the monopolist only cares about the discounted sum of prices that the buyer pays. Since this is a setting with full commitment, the revelation principle holds, and we can focus on direct and truthful mechanisms. Thus, the monopolist's problem is

$$\max \sum_{t=1}^{T} \delta^t (\mu p_t(\bar{b}) + (1-\mu) p_t(\underline{b}))$$

subject to

$$\sum_{t=1}^{T} \delta^t (\bar{b} q_t(\bar{b}) - p_t(\bar{b})) \geq \sum_{t=1}^{T} \delta^t (\bar{b} q_t(\underline{b}) - p_t(\underline{b}))$$

and

$$\sum_{t=1}^{T} \delta^t (\underline{b} q_t(\underline{b}) - p_t(\underline{b})) \geq 0$$

It is obviously optimal to set $p_t(\underline{b}) = \underline{b} q_t(\underline{b})$ and $\bar{b} q_t(\bar{b}) - p_t(\bar{b}) = \bar{b} q_t(\underline{b}) - p_t(\underline{b})$ in each period. Plugging these in, we get that the seller always wants to maximize $q_t(\bar{b})$ and sets $q_t(\underline{b}) = 0$ if $\mu > \frac{\underline{b}}{\bar{b}}$.

□

Lemma A.1 *In any period of any equilibrium of the spot contracting game, the price charged is at least $\underline{b}$.*

Proof Sequential rationality requires that a buyer with a low valuation always expects to receive weakly positive surplus in the future, and is thus receiving strictly positive surplus if the price is lower than $\underline{b}$. Furthermore, both high types and low types will buy at this price, and beliefs will be the same in the next period. Thus, in any period t in which the price is below $\underline{b}$, the monopolist can increase the price to $\underline{b}$ and both high and low valuation consumers will still buy, increasing profits. □

Lemma 1 *In any any period of any equilibrium of the spot contracting game, the seller either*

1. *fully pools with both types buying,*
2. *separates such that high types buy and low types mix between buying and not buying, or*
3. *separates such that low types do not buy and high types buy or mix between buying and not buying.*

Proof In a given period, there will be at most two options for the buyer to choose from: buying at a price weakly greater than $\underline{b}$ or not buying. For a given price offer, each type can either not buy (N), mix between buying and not buying (M), or buy with probability 1 (B). Thus, all of the possibilities in a given period can be described by $\{\underline{N}, \underline{M}, \underline{B}\} \times \{\bar{N}, \bar{M}, \bar{B}\}$, where the underline describes the behavior of low types and the bar describes behavior of high types. The lemma then claims that we can focus on $(\underline{B}, \bar{B})$, $(\underline{M}, \bar{B})$, $(\underline{N}, \bar{M})$, and $(\underline{N}, \bar{B})$.

The seller will not fully pool both types not buying because she can strictly increase profits by selling in the current period to both types at price $\underline{b}$ and using the same continuation equilibrium. Neither type's incentive constraints are violated. This rules out $(\underline{N}, \bar{N})$.

There cannot be an equilibrium in which some low types are buying and no high types are buying. Suppose that there were. In this case the price being charged must be no higher than $\underline{b}$ if the low types are purchasing. The posterior after observing the buyer purchase in that period would be 0, leading to a price of $\underline{b}$ in all future periods. Thus, a high type could receive $\bar{b} - \underline{b}$ in all periods, which is his maximal payoffs, and he would want to purchase. This rules out $(\underline{B}, \bar{N})$ and $(\underline{M}, \bar{N})$.

Suppose that some high types are buying and all low types are buying. Then when the seller observes a buyer not purchase, she knows he is a high types and will charge price $\bar{b}$ in all future periods, leading to 0 surplus. Since low types are buying, the price can be no higher than $\underline{b}$ so high types get strictly positive surplus from buying in the current period. Thus, all high types would purchase in the current period. This rules out $(\bar{B}, \underline{M})$.

The seller also will not separate the buyers into two groups both of which have posteriors strictly between 0 and 1 in period $t + 1$, which is case $(\underline{M}, \bar{M})$. Suppose that this were the case. In this situation, both types must be indifferent between consuming and not consuming in period t. For the buyers with a low valuation, this implies that the price charged to consume in period t is $\underline{b}$, since low types never have a strictly positive continuation value. Thus, high types receive a surplus of $\bar{b} - \underline{b}$ for consuming in the current period. Since the high type is also indifferent between buying and not buying in period t, the difference between the high type's continuation value for these two actions must be $\bar{b} - \underline{b}$.

In any period the low types do not buy, the monopolist maximizes profits by charging a price which makes the high types indifferent between buying and not buying. This implies that the high type's continuation value must be equal to the discounted sum of low type consumption multiplied by $(\bar{b} - \underline{b})$. Thus, if the difference between the two possible continuation values for the high types is

$\bar{b}-\underline{b}$, then the difference in discounted consumption for the low types must be equal to 1. However, Assumption 1 guarantees that there is no sequence of purchases which fulfills this requirement, so the seller cannot be separating the buyers into two groups with posteriors strictly between 0 and 1. This rules out $(\underline{M}, \bar{M})$. $\square$

Proposition 3 *In the optimal equilibrium of the spot contracting game there exist numbers* $\underline{\mu}_t$ *and* $\hat{\mu}_t$, $0 < \underline{\mu}_t \leq \hat{\mu}_t < 1$ *such that*

- *for* $\mu \geq \hat{\mu}_t$ *the low types do not buy and the high types either mix or buy with probability one,*
- *for* $\mu \in (\underline{\mu}_t, \hat{\mu}_t)$ *either both types buy or the high types buy and the low types mix,*
- *for* $\mu \leq \underline{\mu}_t$ *both types buy.*

To prove Proposition 3 I will first prove a series of lemmas before showing the main result. Define $V_{t+1}(\mu)$ as the seller's optimal payoffs starting in period $t+1$ as a function of beliefs in period $t+1$ and $W_{t+1}(\mu)$ as the correspondence between beliefs in period $t+1$ and all possible expected payoffs for high valuation buyers starting in period $t+1$ given that the seller is optimizing payoffs.

Lemma 1 shows that there are three possible outcomes in any period. I will show the properties of the seller's and high-type buyer's payoffs conditional on carrying out each of these outcomes. Thus, given V_{t+1} and W_{t+1}, define $V_t^{\mathrm{FP}}(\mu)$ and $W_t^{\mathrm{FP}}(\mu)$ as the payoffs to the seller and buyer when the seller is optimally fully pooling buyers, $V_t^{\mathrm{LM}}(\mu)$ and $W_t^{\mathrm{LM}}(\mu)$ as the payoffs to the seller and buyer when the seller is optimally causing low types to mix, and $V_t^{\mathrm{HM}}(\mu)$ and $W_t^{\mathrm{HM}}(\mu)$ as the payoffs to the seller and buyer when the seller is optimally causing high types to mix.

Lemma A.2 *Suppose that* V_{t+1} *is increasing, piecewise linear, and convex,* W_{t+1} *is a correspondence which a step function on the domain where it is single valued and with the set values equal to the convex hull of the two "step" values, and that the points at which* W_{t+1} *has multiple values are the points where* V_{t+1} *changes slope. Then*

$$V_t^{FP}(\mu) = \underline{b} + \delta V_{t+1}(\mu)$$
$$W_t^{FP}(\mu) = \bar{b} - \underline{b} + \delta W_{t+1}(\mu).$$

Proof Since all buyers receive the same allocation, they receive the same price which is optimally set at $\underline{b}$ to satisfy the low valuation buyer's individual rationality constraint. Beliefs do not change going into the next period. The seller receives her discounted continuation value plus the price $\underline{b}$ and high valuation buyers receive their discounted continuation value plus the difference between their consumption value and the price. $\square$

Lemma A.3 *Suppose that V_{t+1} is increasing, piecewise linear, and convex, W_{t+1} is a correspondence which a step function on the domain where it is single valued and with the set values equal to the convex hull of the two "step" values, and that the points at which W_{t+1} has multiple values are the points where V_{t+1} changes slope. Then*

- *$V_t^{LM}(\mu)$ intersects V_t^{FP} at one of the points at which it is kinked, and is equal to $-\infty$ above this point,*
- *$V_t^{LM}(\mu)$ is linear below the point it intersects V_t^{FP} and has an intercept of $V_{t+1}(0)$,*
- *W_t^{LM} is equal to $-\infty$ where $V_t^{LM}(\mu) = \infty$, and is otherwise a constant function equal to*

$$\bar{b} - \underline{b} + \delta \max \left\{ W_{t+1} \left(\frac{\mu_t}{1 - \underline{x}_t^*(1-\mu_t)} \right) \right\}$$

for the optimal low type mixing rate $\underline{x}_t^*$.

Proof When optimizing conditional on high types always buying and low types mixing between buying and not buying, the seller must charge a price of $\underline{b}$ to satisfy the low valuation type's incentive compatibility constraint. The seller's problem can be written as

$$\max_{\underline{x}_t} \ \underline{x}_t(1-\mu_t)\delta V_{t+1}(0) + (1 - \underline{x}_t(1-\mu_t)) \left[\underline{b} + \delta V_{t+1} \left(\frac{\mu_t}{1 - \underline{x}_t(1-\mu_t)} \right) \right]$$

subject to

$$\bar{b} - \underline{b} + \max \left\{ W_{t+1} \left(\frac{\mu_t}{1 - \underline{x}_t(1-\mu_t)} \right) \right\} \geq W_{t+1}(0).$$

The low type's incentive compatibility constraint is fulfilled because the price being charged is $\underline{b}$, and the remaining constraint is the high type's incentive compatibility constraint.

Here, the objective function is piecewise linear and convex due to the discontinuities in the slopes of V_{t+1}. Thus, the optimal mixing rate is either $\underline{x}_t = 0$ (which is equivalent to the full pooling case) or setting $\underline{x}_t$ equal to the maximum value which does not violate the high type's incentive compatibility constraints.

Since I am focused on the case in which the low type actually mixes, $V_t^{\mathrm{LM}}(\mu)$ is the payoffs the seller receives from setting $\underline{x}_t$ equal to the maximum value which does not violate the high type's incentive compatibility constraints. This value does not always exist, so set $V_t^{\mathrm{LM}} = -\infty$ if it does not. Then where it exists, V_t^{LM} is linear with an intercept of $V_{t+1}(0)$. Since the optimal allocation is causing the low type to mix between revealing their valuation and pooling with the high types by purchasing at price $\underline{b}$, V_t^{LM} intersects $V_t^{\mathrm{FP}}(\mu_t)$ at the maximum value for which V_t^{LM} is positive.

W_t^{LM} describes the payoffs the high valuation buyer receives for this optimal mixing rule. I define $W_t^{\mathrm{LM}} = -\infty$ where $V_t^{\mathrm{LM}} = -\infty$. When the problem is feasible, high valuation buyers consume at price $\bar{b} - \underline{b}$, and the continuation equilibrium involves promising the maximum amount possible to the high type conditional on beliefs in the next period (in order to relax the high type's incentive compatibility constraint. Thus, on the region it is greater than $-\infty$,

$$W_t^{\mathrm{LM}} = \bar{b} - \underline{b} + \delta \max \left\{ W_{t+1} \left(\frac{\mu_t}{1 - \underline{x}_t^* (1 - \mu_t)} \right) \right\}$$

where $\underline{x}_t^*$ is the optimal mixing rate. $\square$

Lemma A.4 *Suppose that V_{t+1} is increasing, piecewise linear, and convex, W_{t+1} is a correspondence which a step function on the domain where it is single valued and with the set values equal to the convex hull of the two "step" values, and that the points at which W_{t+1} has multiple values are the points where V_{t+1} changes slope. Then*

- *$V_t^{HM}(\mu_t)$ can be equal to $-\infty$ below some cutoff, but is otherwise increasing, piecewise linear, and convex,*
- *$W_t^{HM}(\mu_t) = -\infty$ where $V_t^{HM}(\mu_t) = -\infty$, but otherwise takes the same form as W_{t+1}.*

Proof When optimizing conditional on low types not buying and high types buying with probability greater than 0, the seller's problem amounts to choosing a price which will determine the proportion of high types that purchase in period t. This problem can be written as

$$\max_{p_t, \bar{x}_t} \; (\bar{x}_t \mu_t) \left[p_t + \delta V_{t+1}(1) \right] + (1 - \bar{x}_t \mu_t) \, \delta V_{t+1} \left(\frac{(1 - \bar{x}_t) \mu_t}{1 - \bar{x}_t \mu_t} \right)$$

subject to

$$\begin{aligned} & p_t \leq \bar{b} - \delta \min \left\{ W_{t+1} \left(\frac{(1 - \bar{x}_t) \mu_t}{1 - \bar{x}_t \mu_t} \right) \right\} \\ & p_t \geq \underline{b} \\ & \bar{x}_t \in [0, 1] \end{aligned}$$

where $\bar{x}_t$ is the probability that the high valuation types buy. The first constraint is due to the fact that to buy, high value buyers must receive a payoff which is weakly higher than what they would receive if they did not buy in the current period and instead imitated low valuation buyers. This must hold with equality if the high valuation buyer is mixing strictly. The second constraint is the "reverse incentive compatibility constraint" which arises from Lemma A.1. Since the seller's objective function is increasing in p_t, this problem can be rewritten as

$$\max_{\bar{x}_t} \ (\bar{x}_t\mu_t)\left[\bar{b} - \delta \min\left\{W_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)\right\} + \delta V_{t+1}(1)\right] + (1-\bar{x}_t\mu_t)\,\delta V_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right) \quad (1)$$

subject to

$$\bar{b} - \delta \min\left\{W_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)\right\} \geq \underline{b}$$

$$\bar{x}_t \in [0,1].$$

Where this problem has a solution, define the seller's profits as $V_t^{\mathrm{HM}}(\mu_t)$ (otherwise, let $V_t^{\mathrm{HM}}(\mu_t) = -\infty$).

Problem (1) does not have a solution when $W_{t+1}(\mu_t) > \frac{\bar{b}-\underline{b}}{\delta}$. When this is the case, the price necessary to incentivize high valuation buyers to reveal their types is strictly lower than $\underline{b}$, which contradicts Lemma A.1.

The assumptions on V_{t+1} and W_{t+1} imply that the objective function is almost everywhere increasing in $\bar{x}_t$, but decreases discontinuously at the points where $\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}$ is equal to one of the points where W_{t+1} is set valued. Thus, if the seller chooses prices such that no low types purchase in period t, the beliefs in period $t+1$ will be one of the points at which the slope of V_{t+1} changes.

Consider two potential posteriors in period $t+1$: μ and μ'. For a fixed posterior in the next period, Problem (1) shows that payoffs starting from the current period are linear in current beliefs. Furthermore, if $\mu < \mu'$, simple algebra shows that the slope of payoffs as a function of current beliefs is higher when $\mu_{t+1} = \mu'$, and the payoffs must cross between μ' and 1. Thus, on the domain where there is a solution to the problem, $V_t^{\mathrm{HM}}(\mu_t)$ is increasing, piecewise linear, and convex.

Again, set $W_t^{\mathrm{HM}}(\mu_t) = -\infty$ if the constraints lead Problem (1) to not have a solution. I showed above that higher values of μ_t lead to weakly higher posterior beliefs. Since continuation values in period t are simply the discounted continuation values from period $t+1$, this implies that W_t^{HM} is weakly increasing. Furthermore, on each interval in which $V_t^{\mathrm{HM}}(\mu_t)$ is linear, the posteriors are constant so W_t^{HM} must also be constant. Thus, W_t^{HM} has the same structure as W_{t+1}. □

Lemma A.5 *Suppose that V_{t+1} is increasing, piecewise linear, and convex, W_{t+1} is a correspondence which a step function on the domain where it is single valued and with the set values equal to the convex hull of the two "step" values, and that the points at which W_{t+1} has multiple values are the points where V_{t+1} changes slope. Then there exist numbers $\underline{\mu}_t$ and $\hat{\mu}_t$, $0 < \underline{\mu}_t \leq \hat{\mu}_t < 1$ such that*

$$V_t(\mu) = \begin{cases} V_t^{FP}(\mu) \ \textit{if} \ \mu \leq \underline{\mu}_t \\ \max\{V_t^{FP}(\mu), V_t^{LM}(\mu)\} \ \textit{if} \ \underline{\mu}_t < \mu < \hat{\mu}_t \\ V_t^{HM}(\mu) \ \textit{otherwise} \end{cases}$$

$$W_t(\mu) = \begin{cases} W_t^{FP}(\mu) \ \textit{if} \ \mu \leq \underline{\mu}_t \\ W_t^{FP}(\mu) \ \textit{if} \ \underline{\mu}_t < \mu < \hat{\mu}_t \ \textit{and} \ V_t^{FP}(\mu) \geq V_t^{LM}(\mu) \\ W_t^{LM}(\mu) \ \textit{if} \ \underline{\mu}_t < \mu < \hat{\mu}_t \ \textit{and} \ V_t^{LM}(\mu) > V_t^{FP}(\mu) \\ W_t^{HM}(\mu) \ \textit{otherwise} \end{cases}$$

Proof of Lemma A.5 The payoffs that the seller receives in a given period are equal to the highest payoffs from either fully pooling, optimally allowing high types to mix, or optimally allowing low types to mix. Thus,

$$V_t(\mu) = \max\{V_t^{\mathrm{FP}}(\mu), V_t^{\mathrm{LM}}(\mu), V_t^{\mathrm{HM}}(\mu)\},$$

This always exists on the unit interval because V_t^{FP} always exists on the unit interval.

First, I will show that $V_t^{\mathrm{FP}}(\mu_t)$ intersects $V_t^{\mathrm{HM}}(\mu_t)$ exactly once and crosses from below.

It is clear that $V_t^{\mathrm{FP}}(1) = V_{t+1}(1) + \underline{b} < V_{t+1}(b) + \bar{b} = V_t^{\mathrm{HM}}(1)$. Furthermore, Lemma A.4 shows that for the minimum $\hat{\mu}$ such that the high type mixing problem has a solution, $V_t^{\mathrm{FP}}(\hat{\mu}) = V_{t+1}(\hat{\mu}) + \underline{b} > V_{t+1}(\hat{\mu}) = V_t^{\mathrm{HM}}(\hat{\mu})$. Thus, $V_t^{\mathrm{FP}}(\mu_t)$ and $V_t^{\mathrm{HM}}(\mu_t)$ cross at least once.

Suppose that $V_t^{\mathrm{FP}}(\mu_t)$ and $V_t^{\mathrm{HM}}(\mu_t)$ cross more than once. Lemmas A.2 and A.4 show that both functions are piecewise linear so, there exist two kink points of $V_t^{\mathrm{HM}}(\mu_t)$, $\hat{\mu}$ and $\hat{\mu}'$ such that $\hat{\mu} < \hat{\mu}'$, $V_t^{\mathrm{FP}}(\hat{\mu}) \leq V_t^{\mathrm{HM}}(\hat{\mu})$, but $V_t^{\mathrm{FP}}(\hat{\mu}') > V_t^{\mathrm{HM}}(\hat{\mu}')$. Then

$$\begin{aligned} V_t^{\mathrm{HM}}(\hat{\mu}') &= \left(\frac{1-\hat{\mu}'}{1-\hat{\mu}}\right) V_t^{\mathrm{HM}}(\hat{\mu}) + \left(\frac{\hat{\mu}'-\hat{\mu}}{1-\hat{\mu}}\right)\left[p_t^{\mathrm{HM}}(\hat{\mu}) + \delta V_{t+1}(1)\right] \\ &\geq \left(\frac{1-\hat{\mu}'}{1-\hat{\mu}}\right) V_t^{\mathrm{HM}}(\hat{\mu}) + \left(\frac{\hat{\mu}'-\hat{\mu}}{1-\hat{\mu}}\right)\left[\underline{b} + \delta V_{t+1}(1)\right] \\ &\geq \left(\frac{1-\hat{\mu}'}{1-\hat{\mu}}\right) V_t^{\mathrm{FP}}(\hat{\mu}) + \left(\frac{\hat{\mu}'-\hat{\mu}}{1-\hat{\mu}}\right)\left[\underline{b} + \delta V_{t+1}(1)\right] \\ &\geq V_t^{\mathrm{FP}}(\hat{\mu}') \end{aligned}$$

where $p_t^{\mathrm{HM}}(\hat{\mu})$ is the optimal price to charge when high types are mixing in period t and beliefs are $\hat{\mu}$. The first equality takes advantage of the piecewise linearity of the function V_t^{HM}. The inequality in the second line uses the fact that $p_t^{\mathrm{HM}}(\hat{\mu}) \geq \underline{b}$. The inequality in the third line holds because the maintained assumption is that $V_t^{\mathrm{FP}}(\hat{\mu}) \leq V_t^{\mathrm{HM}}(\hat{\mu})$. The final inequality holds because $V_t^{\mathrm{FP}}(1) = \underline{b} + \delta V_{t+1}(1)$ and V_t^{FP} is convex. The contradicts the assumption that $V_t^{\mathrm{FP}}(\hat{\mu}') > V_t^{\mathrm{HM}}(\hat{\mu}')$, so $V_t^{\mathrm{FP}}(\mu_t)$ and $V_t^{\mathrm{HM}}(\mu_t)$ cross a single time.

Lemma A.3 shows that V_t^{LM} is linear on the region it is greater than $-\infty$. Because of this, it intersects V_t^{HM} at most once. Thus, we can take the highest value at which V_t^{HM} intersects either V_t^{FP} or V_t^{LM}. Above this value, it is optimal for the seller to make high valuation buyers mix and low valuation buyers not buy. Below this cutoff, it is optimal to use one of the other two options. This defines $\hat{\mu}_t$.

Next consider $V_t^{\mathrm{LM}}(\mu_t)$. Lemma A.3 shows that $V_t^{\mathrm{LM}}(0) < V_t^{\mathrm{FP}}(0)$, that V_t^{LM} is linear on the region it is greater than $-\infty$, and that it is equal to V_t^{FP} at the maximum value for which it is greater than $-\infty$. This implies that V_t^{LM} intersects V_t^{FP} either one or two times. If there are two intersection points and the first is lower than $\hat{\mu}_t$, then $\underline{\mu}_t$ is equal to this point. Otherwise, $\underline{\mu}_t = \hat{\mu}_t$.
□

We can now prove Proposition 3.

Proof of Proposition 3 The proof will use backwards induction starting from period T.

Basis Step: Notice first that in period T, the monopolist faces a single period screening problem and thus chooses prices to maximize profits in that period given μ_T. To do so, she sets a price of $\underline{b}$ and sells to both the high types and the low types if $\mu_T < \frac{\underline{b}}{\bar{b}}$ or sets a price of $\bar{b}$ and sells only to the remaining unidentified high types if $\mu_T > \frac{\underline{b}}{\bar{b}}$. When $\mu_T = \frac{\underline{b}}{\bar{b}}$ the seller is indifferent between these two options.

Consider the payoffs to each type of player as a function of μ_T. Buyers with valuation $\underline{b}$ always receive payoffs equal to 0. For $\mu_T \in [0, \frac{\underline{b}}{\bar{b}})$, the seller receives payoffs equal to $\underline{b}$ while the high valuation buyer receives payoffs equal to $\bar{b} - \underline{b}$. For $\mu_T \in (\frac{\underline{b}}{\bar{b}}, 1]$, the seller receives payoffs equal to $\mu_T \bar{b}$ while high valuation buyers receive payoffs of 0. When $\mu_T = \frac{\underline{b}}{\bar{b}}$, the seller receives $\underline{b}$ and the buyer can receive any payoff between 0 and $\bar{b} - \underline{b}$. Thus, V_T is increasing, piecewise linear, and convex. W_T is a correspondence and on the domain where it is single valued is a decreasing step function, with the set values equal to the convex hull of the two "step" values. The points at which W_T has multiple values are the points where V_T changes slope.

Inductive Step: Now suppose that V_{t+1} is increasing, piecewise linear, and convex, W_{t+1} is a correspondence which a step function on the domain where it is single valued and with the set values equal to the convex hull of the two "step" values, and that the points at which W_{t+1} has multiple values are the points where V_{t+1} changes slope.

Lemma A.5 shows that under these conditions, it is always optimal to fully pool when beliefs are below some cutoff and to have only high types buying with positive probability when beliefs are above the cutoff, with either fully pooling or low types mixing between the two cutoffs. Thus, I need only prove that when this is the case, V_t and W_t satisfy the necessary properties.

In the construction of V_t in Lemma A.5, I showed that it is the upper envelope of three functions and that the functions "overtake" each other by intersection from below. Because where they are greater than $-\infty$ the functions are all increasing, convex, and piecewise linear, V_t itself is also increasing, convex, and piecewise linear.

On the interior of the regions where one of the three V_t^i functions dominate, W_t inherits the properties of the the respective high type payoff correspondence, which always satisfies the necessary requirements. Furthermore, when one of the V_t^i functions overtakes the other, the seller is indifferent between the two options and can randomize between them, implying that she can implement any payoff for the high type buyer in the convex hull of the two payoffs. Thus, I need only show that payoffs are weakly decreasing at these intersection points, which could include one of four possibilities: 1) LM payoffs overtaking FP payoffs, 2) FP payoffs overtaking LM payoffs, 3) HM payoffs overtaking FP payoffs, or 4) HM payoffs overtaking LM payoffs.

Lemma A.3 shows that where it is greater than $-\infty$, W_t^{LM} is equal to the value of W_t^{FP} at the point where V_t^{LM} and V_t^{LM} intersect. This implies that W_t is continuous when FP payoffs overtake LM payoffs, which is possibility 2 above. Furthermore, since W_t^{FP} is weakly decreasing, this also implies that W_t^{LM} is strictly less than W_t^{FP} where LM payoffs overtake FP payoffs, which is possibility 1 above.

Notice that in the full pooling and low mixing cases, the high valuation buyer receives the good at price $\underline{b}$, thus receiving payoffs $\bar{b} - \underline{b}$ plus some weakly positive continuation value. When only high types are mixing, the high types payoffs are equal to $\bar{b}$ minus the price, which is always greater than $\underline{b}$. Thus, payoffs for high types are always higher when there is full pooling or low types mixing than when the high types are mixing, and payoffs strictly decrease when HM payoffs overtake FP or LM payoffs (possibilities 3 and 4 above).

Thus, V_t and W_t satisfy the conditions we assumed on V_{t+1} and W_{t+1}, completing the inductive step. $\square$

Corollary 1 *Low types mix in period t only if $T - t \geq 3$.*

Proof The proof of Proposition 3 showed that where they are non-negative, the payoffs arising from low types mixing V_t^{LM} are linear with $V_t^{\text{LM}}(0) < V_t^{\text{FP}}(0)$. Furthermore, these payoffs intersect

V_t^{FP} at one of the interior points at which V_t^{FP} is kinked and are equal to $-\infty$ beyond that point. Because V_t^{FP} is piecewise linear, increasing, and convex, if $V_t^{\text{LM}}(\mu) > V_t^{\text{FP}}(\mu)$ for any μ, then $V_t^{\text{LM}}(\mu) > V_t^{\text{FP}}(\mu)$ at one of the points at which $V_t^{\text{FP}}(\mu)$ is kinked.

At the lowest kink point of V_t^{FP}, the seller is receiving the same payoffs as the full commitment equilibrium, since fully pooling is optimal in every period in this case. Thus, if $V_t^{\text{LM}}(\mu) > V_t^{\text{FP}}(\mu)$ at one of the points at which $V_t^{\text{FP}}(\mu)$ is kinked, then $V_t^{\text{FP}}(\mu)$ must have at least three kinks.

Because $V_t^{\text{FP}}(\mu)$ is simply a linear transformation of $V_{t+1}(\mu)$, they both have the same number of kinks. Thus, if $V_t^{\text{LM}}(\mu) > V_t^{\text{FP}}(\mu)$ for any μ then V_{t+1} must have at least three kinks. However, V_T and V_{T-1} always have one and two kinks respectively, so $t+1 < T-1$, giving the result. $\square$

B Strategies in Examples

B.1 Example 1

In example 1 we have $\underline{b} = 1$, $\bar{b} = 3$, $\delta = 0.6$, and $T = 4$.

B.1.1 Seller's Strategy

The seller's strategy in each period is to offer a (potentially random) price in each period as a function of the history. The price offered in period t will be defined as p_t, the history in period t as h^t, and the beliefs as μ_t. The randomizations that are being used at certain beliefs are designed to make the buyer strictly prefer randomizing with the appropriate probability in earlier periods. I write the strategies in terms of primitive parameters ($\bar{b}$, $\underline{b}$, and δ) in order to make the reason for the strategies clearer, but the form of the strategies is not invariant to the specific values of the parameters that I have assumed.

$$p_4(h^4) = \begin{cases} \bar{b} \text{ if } \mu_4 > \frac{1}{3} \\ (\bar{b}, \underline{b}) \text{ with prob. } (\pi_{4,1}, 1-\pi_{4,1}) \text{ if } \mu_4 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_3(h^3) = \begin{cases} \bar{b} \text{ if } \mu_3 > \frac{7}{11} \\ (\bar{b}, \bar{b} - (\bar{b} - \underline{b})\delta) \text{ with prob. } (\pi_{3,2}, 1-\pi_{3,2}) \text{ if } \mu_3 = \frac{7}{11} \\ \bar{b} - (\bar{b} - \underline{b})\delta \text{ if } \frac{7}{11} > \mu_3 > \frac{1}{3} \\ (\bar{b} - (\bar{b} - \underline{b})\delta, \underline{b}) \text{ with prob. } (\pi_{3,1}, 1-\pi_{3,1}) \text{ if } \mu_3 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_2(h^2) = \begin{cases} \bar{b} \text{ if } \mu_2 > \frac{167}{191} \\ (\bar{b}, \bar{b} - \delta^2(\bar{b} - \underline{b})) \text{ with prob. } (\pi_{2,3}, 1-\pi_{2,3}) \text{ if } \mu_2 = \frac{167}{191} \\ \bar{b} - \delta^2(\bar{b} - \underline{b}) \text{ if } \frac{167}{191} > \mu_2 > \frac{7}{11} \\ (\bar{b} - \delta^2(\bar{b} - \underline{b}), \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b})) \text{ with prob. } (\pi_{2,2}, 1-\pi_{2,2}) \text{ if } \mu_2 = \frac{7}{11} \\ \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) \text{ if } \frac{7}{11} > \mu_2 > \frac{1}{3} \\ (\bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}), \underline{b}) \text{ with prob. } (\pi_{2,1}, 1-\pi_{2,1}) \text{ if } \mu_2 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_1 = \begin{cases} \bar{b} \text{ if } \mu > \frac{18631}{19063} \\ \bar{b} - \delta^3(\bar{b} - \underline{b}) \text{ if } \frac{18631}{19063} \geq \mu > \frac{167}{191} \\ \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \text{ if } \frac{167}{191} \geq \mu > \frac{3507}{5851} \\ \underline{b} \text{ if } \frac{3507}{5851} \geq \mu \end{cases}$$

where $\pi_{2,1} = \max\{\min\{1 + \delta + \delta^2 - \frac{\bar{b} - p_1}{\delta(\bar{b} - \underline{b})}, 1\}, 0\}$, $\pi_{2,2} = \max\{\min\{1 + \delta - \frac{\bar{b} - p_1}{\delta(\bar{b} - \underline{b})}, 1\}, 0\}$, $\pi_{2,3} = \max\{\min\{1 - \frac{\bar{b} - p_1}{\delta^3(\bar{b} - \underline{b})}, 1\}, 0\}$, $\pi_{3,1} = \max\{\min\{1 + \delta - \frac{\bar{b} - p_2}{\delta(\bar{b} - \underline{b})}, 1\}, 0\}$, $\pi_{3,2} = \max\{\min\{1 - \frac{\bar{b} - p_2}{\delta^2(\bar{b} - \underline{b})}, 1\}, 0\}$, and $\pi_{4,1} = \max\{\min\{1 - \frac{\bar{b} - p_3}{\delta(\bar{b} - \underline{b})}, 1\}, 0\}$.

Value functions:

$$V_4(\mu_4) = \begin{cases} 3\mu_4 \text{ if } \mu_4 > \frac{1}{3} \\ 1 \text{ otherwise} \end{cases}$$

$$V_3(\mu_3) = \begin{cases} 6.3\mu_t - 1.5 \text{ if } \mu_3 > \frac{7}{11} \\ 3\mu_t + 0.6 \text{ if } \frac{7}{11} \geq \mu_3 > \frac{1}{3} \\ 1.6 \text{ otherwise} \end{cases}$$

$$V_2(\mu_2) = \begin{cases} 12.03\mu_2 - 6.15 \text{ if } \mu_2 > \frac{167}{191} \\ 6.3\mu_2 - 1.14 \text{ if } \frac{167}{191} \geq \mu_2 > \frac{7}{11} \\ 3\mu_2 + 0.96 \text{ if } \frac{7}{11} \geq \mu_2 > \frac{1}{3} \\ 1.96 \text{ otherwise} \end{cases}$$

$$V_1(\mu) = \begin{cases} \frac{31093}{1000}\mu - \frac{24565}{1000} \text{ if } \mu > \frac{18631}{19063} \\ 12.03\mu - \frac{2967}{500} \text{ if } \frac{18631}{19063} \geq \mu > \frac{167}{191} \\ 6.3\mu - 0.924 \text{ if } \frac{167}{191} \geq \mu > \frac{3507}{5851} \\ \frac{467}{167}\mu + 1.176 \text{ if } \frac{3507}{5851} \geq \mu > \frac{167}{416} \\ 1.8\mu + 1.576 \text{ if } \frac{167}{416} \geq \mu > \frac{1}{3} \\ 2.176 \text{ otherwise} \end{cases}$$

B.1.2 Low Valuation Buyer's Strategy

$$\underline{x}_4(h^4) = \begin{cases} 0 \text{ if } \underline{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_3(h^3) = \begin{cases} 0 \text{ if } \underline{b} < p_3 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_2(h^2) = \begin{cases} 0 \text{ if } \underline{b} < p_2 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_1(h^1) = \begin{cases} 0 \text{ if } p_1 > \underline{b} \\ \frac{24\mu}{167-167\mu} \text{ if } p_1 = \underline{b} \text{ and } \mu_1 > \frac{167}{416} \\ 1 \text{ otherwise} \end{cases}$$

B.1.3 High Valuation Buyer's Strategy

$$\bar{x}_4(h^4) = \begin{cases} 0 \text{ if } \bar{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_3 < \frac{1}{3}$,

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \\ 1 \text{ otherwise} \end{cases}$$

otherwise if $\mu_3 \geq \frac{1}{3}$

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} < p_3 \\ \frac{3\mu_3 - 1}{2\mu_3} \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \leq \bar{b} \\ 1 \text{ otherwise} \end{cases} .$$

If $\mu_2 < \frac{1}{3}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_4 \\ 1 \text{ otherwise} \end{cases} ,$$

while if $\frac{1}{3} \leq \mu_2 < \frac{7}{11}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \\ \frac{3\mu_2 - 1}{2\mu_2} \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{7}{11} \leq \mu_2$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} < p_2 \\ \frac{11\mu_2 - 7}{4\mu_2} \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \leq \bar{b} \\ \frac{3\mu_2 - 1}{2\mu_2} \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

If $\mu < \frac{1}{3}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \underline{b} < p_1 \\ 1 \text{ otherwise} \end{cases},$$

if $\frac{1}{3} \leq \mu < \frac{7}{11}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

if $\frac{7}{11} \leq \mu < \frac{167}{191}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \\ \frac{11\mu - 7}{4\mu} \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{167}{191} \leq \mu$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} < p_1 \\ \frac{191\mu - 167}{24\mu} \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \leq \bar{b} \\ \frac{11\mu - 7}{4\mu} \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

B.2 Example 2

In Example 2 we have $\underline{b} = 1$, $\bar{b} = 2$, $\delta = 0.7$, and $T = 4$.

B.2.1 Seller's Strategy

The seller's strategy in each period is to offer a (potentially random) price in each period as a function of the history. The price offered in period t will be defined as p_t, the history in period t as h^t, and the beliefs as μ_t. The randomizations that are being used at certain beliefs are designed to make the buyer strictly prefer randomizing with the appropriate probability in earlier periods. I write the strategies in terms of primitive parameters ($\bar{b}$, $\underline{b}$, and δ) in order to make the reason for the strategies clearer, but the form of the strategies is not invariant to the specific values of the parameters that I have assumed.

$$p_4(h^4) = \begin{cases} \bar{b} \text{ if } \mu_4 > 0.5 \\ (\bar{b}, \underline{b}) \text{ with prob. } (\pi_{4,1}, 1-\pi_{4,1}) \text{ if } \mu_4 = \frac{1}{2} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_3(h^3) = \begin{cases} \bar{b} \text{ if } \frac{27}{34} < \mu_3 \\ \left(\bar{b}, \bar{b} - (\bar{b} - \underline{b})\delta\right) \text{ with prob. } (\pi_{3,2}, 1-\pi_{3,2}) \text{ if } \mu_3 = \frac{27}{34} \\ \bar{b} - (\bar{b} - \underline{b})\delta \text{ if } \frac{1}{2} < \mu_3 < \frac{27}{34} \\ (\bar{b} - (\bar{b} - \underline{b})\delta, \underline{b}) \text{ with prob. } (\pi_{3,1}, 1-\pi_{3,1}) \text{ if } \mu_3 = \frac{1}{2} \\ \underline{b} \text{ if } \mu_3 < \frac{1}{2} \end{cases}$$

$$p_2(h^2) = \begin{cases} \bar{b} \text{ if } \frac{5323}{5666} < \mu_2 \\ \left(\bar{b}, 1 - (\bar{b} - \underline{b})\delta^2\right) \text{ with prob. } (\pi_{2,2}, 1-\pi_{2,2}) \text{ if } \mu_2 = \frac{5323}{5666} \\ \bar{b} - (\bar{b} - \underline{b})\delta^2 \text{ if } \frac{3}{4} < \mu_2 < \frac{5323}{5666} \\ \left(\bar{b} - (\bar{b} - \underline{b})\delta^2, \underline{b}\right) \text{ with prob. } (\pi_{2,1}, 1-\pi_{2,1}) \text{ if } \mu_2 = \frac{3}{4} \\ \underline{b} \text{ if } \mu_2 < \frac{3}{4} \end{cases}$$

$$p_1(h^1) = \begin{cases} \bar{b} \text{ if } \frac{10413789}{10531438} \leq \mu \\ \bar{b} - (\bar{b} - \underline{b})\delta^3 \text{ if } \frac{7}{8} \leq \mu < \frac{10413789}{10531438} \\ \underline{b} \text{ if } \mu < \frac{7}{8} \end{cases}$$

where $\pi_{2,1} = \max\left\{\min\left\{\frac{1+\delta+\delta^2}{1+\delta} - \frac{\bar{b}-p_1}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{2,2} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_1}{\delta^3(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{3,1} = \max\left\{\min\left\{1 + \delta - \frac{\bar{b}-p_2}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{3,2} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_2}{\delta^2(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, and $\pi_{4,1} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_3}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$.

Value functions:

$$V_4(\mu_4) = \begin{cases} 2\mu_4 \text{ if } \mu_4 > \frac{1}{2} \\ 1 \text{ otherwise} \end{cases}$$

$$V_3(\mu_3) = \begin{cases} \frac{27}{5}\mu_3 - 2 \text{ if } \mu_3 > \frac{27}{34} \\ 2\mu_3 + 0.7 \text{ if } \frac{27}{34} \geq \mu_3 > \frac{1}{2} \\ 1.7 \text{ otherwise} \end{cases}$$

$$V_2(\mu_2) = \begin{cases} \frac{4723}{350}\mu_2 - \frac{319}{35} \text{ if } \mu_2 > \frac{5323}{5666} \\ 5.4\mu_2 - 1.51 \text{ if } \frac{5323}{5666} \geq \mu_2 > \frac{3}{4} \\ 1.4\mu_2 + 1.49 \text{ if } \frac{3}{4} \geq \mu_2 > \frac{1}{2} \\ \frac{219}{100} \text{ otherwise} \end{cases}$$

$$V_1(\mu) = \begin{cases} \frac{7285989}{171500}\mu_1 - \frac{641717}{17150} \text{ if } \mu > \frac{10413789}{10531438} \\ \frac{589}{50}\mu - \frac{7057}{1000} \text{ if } \frac{10413789}{10531438} \geq \mu > \frac{7}{8} \\ \frac{189}{50}\mu - \frac{57}{1000} \text{ if } \frac{7}{8} \geq \mu > \frac{3}{4} \\ \frac{49}{50}\mu + \frac{2043}{1000} \text{ if } \frac{3}{4} \geq \mu > \frac{1}{2} \\ \frac{2533}{1000} \text{ otherwise} \end{cases}$$

B.2.2 Low Valuation Buyer's Strategy

The low valuation buyer chooses to buy the item in period t if and only if $p_t \leq \underline{b}$.

B.2.3 High Valuation Buyer's Strategy

$$\bar{x}_4(h^4) = \begin{cases} 0 \text{ if } \bar{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_3 < \frac{1}{2}$

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \\ 1 \text{ otherwise} \end{cases}$$

while if $\frac{1}{2} \leq \mu_3$,

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} < p_3 \\ \frac{2\mu_3 - 1}{\mu_3} \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \leq \bar{b} \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_2 < \frac{1}{2}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \underline{b} < p_2 \\ 1 \text{ otherwise} \end{cases},$$

while for $\frac{1}{2} \leq \mu_2 < \frac{27}{34}$, the buyer uses

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \\ \frac{2\mu_2 - 1}{\mu_2} \text{ if } \underline{b} < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{27}{34} \leq \mu_2$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} < p_2 \\ \frac{34\mu_2 - 27}{7\mu_2} \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \leq \bar{b} \\ \frac{2\mu_2 - 1}{\mu_2} \text{ if } \underline{b} < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}.$$

Finally, in the first period, when $\mu_1 < \frac{1}{2}$, the high type's strategy is

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \underline{b} < p_1 \\ 1 \text{ otherwise} \end{cases}$$

For $\frac{3}{4} \leq \mu_1 \leq \frac{5323}{5666}$, the strategy is

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \\ \frac{4\mu - 3}{\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases},$$

and if $\mu_1 > \frac{5323}{5666}$,

$$x_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} < p_1 \\ \frac{5666\mu - 5323}{343\mu} \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \leq \bar{b} \\ \frac{4\mu - 3}{\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases} .$$