Revealed Preference Tests for Linear Probability-Prize Tradeoffs *

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Abstract

We provide necessary and sufficient conditions for expected utility and risk-averse expected utility to rationalize behavior when the decision maker faces linear probability-prize tradeoffs. The setting subsumes those proposed in Andreoni and Harbaugh (2009) and Crosetto and Filippin (2013). The tests are intuitive, straightforward to carry out, and lead naturally to measures for how close the models are to explaining the choice sets. We implement these results empirically and show that only small portions of choice sets can be rationalized by risk-averse expected utility.

JEL classification: C91, D81, D91

Keywords: risk preferences, uncertainty, revealed preference, expected utility, risk aversion.

1 Introduction

Evaluating and measuring risk preferences is a central focus of experimental economics and decision theory. Risk preferences are measured directly as an explanatory variable and indirectly as a potential confound. While researchers often attempt to measure risk aversion using experimental tasks, the revealed preference approach is to instead determine whether choices are consistent with particular models. In this paper, we link the two approaches, showing how to test whether expected utility and risk-averse expected utility explain behavior in a common experimental task used to measure risk aversion.

We study a commonly-used experimental risk elicitation: linear probability-prize tradeoffs. Decision makers must choose from amongst a set of lotteries where increasing the size of the prize decreases the likelihood of receiving that prize by a fixed amount. If the decision maker does not

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receive the prize, they receive nothing. This is a natural setting to elicit risk preferences because subjects are literally trading off "risks" against "rewards." Our research questions are twofold. First, we ask what restrictions a choice set must satisfy in order to be consistent with standard economic models of choice under risk. Second, we empirically investigate how close choice sets from existing experiments are to satisfying these restrictions.

In our theoretical results, we characterize necessary and sufficient conditions for a choice set to be consistent with expected utility (in Theorem 1) and risk-averse expected utility (in Theorem 2). The conditions are straightforward and easy to check in a data set. Our first theoretical result is that the only testable implication of expected utility is that chosen prizes are weakly increasing in the maximum prize available. That is, expected utility maximizers will choose (weakly) higher prizes when we increase the maximum prize. These theoretical results are closely related to Cherchye et al. (2021), which also studies probability-prize tradeoffs. It shows that in a first price auction where the distribution of bids is log-concave, the only implication of expected utility is that bids increase in the bidder's value for the object. Because the linear budgets that we consider in this paper correspond to a uniform (and, thus, log-concave) distribution of bids, this result implies what we find in Theorem 1. Our second theoretical result, Theorem 2, is novel and provides a practical test for consistency with risk-averse expected utility, a model of interest in most applied research. We also show how to compute measurements for *how close* data sets are to being consistent with both expected utility and risk-averse expected utility.

The main contribution of our paper is linking these theoretical results to existing experimental procedures and data sets. We show how to carry out the tests and compute consistency measures using data from Breig and Feldman (2021) for the tasks proposed by Andreoni and Harbaugh (2009). We then complete the same exercise for the "bomb" risk elicitation task (BRET) using data from Crosetto and Filippin (2013). We find that while subjects have moderate levels of consistency with expected utility, only small portions of their choice sets can be rationalized by a risk-averse expected utility function. The reason for this counter-intuitive result is that subjects should always select the same preferred prize from *all* budgets with the same maximal prize if they are risk-averse expected utility maximizers.

Our results can be readily applied to determine whether choice data from commonly used elicitation devices are consistent with standard models. The consistency measures that we propose are easy to compute and will allow researchers to evaluate how close subjects are to choosing according to expected utility.

2 Theory

We study choices from budgets of the form $x + \frac{M_t}{m_t}p = M_t$ for $t = 1, \ldots, T$, where m_t and M_t are strictly positive and x and p are constrained to be weakly positive. The agent selects the combination (x, p) that yields their preferred binary lottery $\{\$x, p; \$0, 1-p\}$. The choice variable x refers to the size of the positive prize that the decision maker will receive, while p refers to the likelihood that they will receive the prize. We define a data set as $\mathcal{O} = \{(p_t, x_t, m_t, M_t)\}_{t=1}^{T}$, where p_t and x_t are the chosen probability and prize when the maximum probability is m_t and the maximum prize is

M_t.

For reasons that will become apparent in our results below, it will be useful to "order" data sets in a particular way. We say that a data set is "regular" if (1) $M_t \geqslant M_{t-1}$ for all $t \geqslant 2$ and (2) $x_{t'} > x_t$ for all $t' > t$ such that $M_t = M_{t'}$. These conditions ensure that the data set is first sorted by M and then by x. Because this amounts to a simple relabeling of the data set and any data set can be relabeled in this way, there is no loss of generality from restricting attention to these regular data sets.

We say that a data set $\mathcal{O}$ is *rationalized by expected utility* if there exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that (1) u is strictly increasing, (2) $u(0) = 0$, and (3) for all t,

$$p_t u(x_t) \geqslant pu(x)$$

for all non-negative p and x such that $x + \frac{M_t}{m_t}p = M_t$. With this definition, we can state how an expected utility maximizer behaves when faced with linear probability-prize tradeoffs.

Theorem 1. *A regular data set $\mathcal{O}$ is rationalized by expected utility if and only if (1) $0 < x_t < M_t$ for all t and (2) $x_t \geqslant x_{t-1}$ for all $t \geqslant 2$.*

In essence, Theorem 1 states that in order for the decision maker's choices to be rationalized by expected utility, it must be the case that they do not make stochastically dominated choices (by choosing the endpoints of the budgets) and that their chosen prize is (weakly) increasing in the maximum prize. Furthermore, the maximum probability available to the decision maker does not matter when determining whether that decision maker chose according to expected utility. Note that expected utility does not impose consistency in choice behavior; that is, an EU decision maker could be indifferent among several options from the same budget and therefore make different choices across repetitions of that budget.

Another implication of Theorem 1 is that it is easy to find the size of the largest proportion of the decision maker's choices that are consistent with expected utility. This can be thought of as an Expected Utility "Houtman-Maks" index, or EU HMI (Houtman and Maks, 1985). When the data set is "regular" as described above, finding (one of) the largest subset(s) of choices that are consistent with expected utility is equivalent to finding the longest increasing subsequence of chosen prizes, conditional on those prizes not being endpoints of the budget. This is straightforward to compute.

While it is common to test for choices being consistent with expected utility, researchers may also be interested in determining whether choices are consistent with *risk-averse* expected utility, which would require a concave utility function.

We say that a data set $\mathcal{O}$ is *rationalized by risk-averse expected utility* if there exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that (1) u is strictly increasing and concave, (2) $u(0) = 0$, and (3) for all t,

$$p_t u(x_t) \geqslant pu(x)$$

for all p and x such that $x + \frac{M_t}{m_t}p = M_t$. With this definition, we can state how a risk-averse expected utility maximizer behaves when faced with linear probability-prize tradeoffs.

Theorem 2. *A regular data set $\mathcal{O}$ is rationalized by a risk-averse expected utility function if and only if (1) $x_1 \leqslant \frac{1}{2}M_1$, (2) $0 < x_t$ for all t, and (3)*

$$0 \leqslant x_t - x_{t-1} \leqslant \frac{1}{2}(M_t - M_{t-1})$$

for all $t \geqslant 2$.

Clearly, in order to satisfy risk-averse expected utility, all of the requirements from Theorem 1 are required to hold (because risk-averse expected utility is a special case of expected utility). However, Theorem 2 has additional requirements. First of all, the choice from the budget with the lowest maximum prize must involve the subject allocating weakly less than half of their budget toward the prize. This is intuitive because allocating exactly half of the budget is the expected-value maximizing (i.e. risk neutral) choice. Second, any other choices must increase the prize by less than half of the change in the maximum prize. A less obvious implication of this is that any choices from budgets that have the same maximum prize must have the same chosen prize (regardless of the maximum probability). This is a very strict requirement that will be discussed further in Section 3.

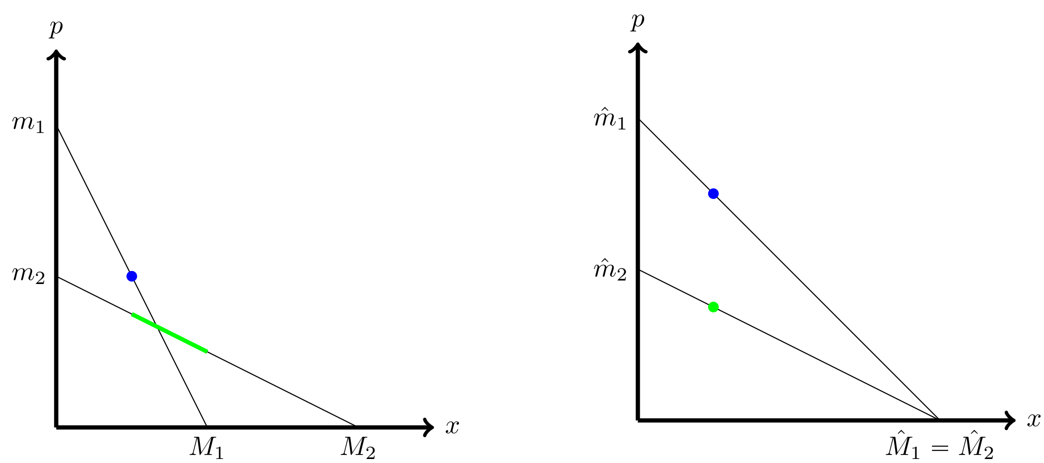

Figure 1: Two pairs of budgets representing of the requirements of risk-averse expected utility. For each pair, we fix an arbitrary choice from the first budget (in blue). The green region in each second budget shows the choices from that budget that would be consistent with risk-averse expected utility. For the pair of budgets on the left, the difference in x from budget 1 to 2 must be between 0 and $\frac{1}{2}(M_2 - M_1)$. For the pair of budgets on the right, the chosen x must be the same from both budgets because $\frac{1}{2}(\hat{M}_2 - \hat{M}_1) = 0$.

Similar to Theorem 1, Theorem 2 has the benefit of making it easy to compute the size of the largest proportion of choices that are consistent with risk-averse expected utility (an index that we call Risk-Averse Expected Utility HMI or RAEU HMI). With a slight restatement, the problem is equivalent to finding the longest path in a directed acyclic graph.[1] Similar to the case of expected

[1] To see this, define each choice that is not an endpoint and spends less than half of its budget on prize as a node in the graph. There is an edge from the node associated with choice i to the node associated with choice j if $j > i$ and $0 \leqslant x_j - x_i \leqslant \frac{1}{2}(M_j - M_i)$. The collection of nodes in any path of such a graph will represent a data set that can be rationalized by risk-averse expected utility. Thus, finding the longest path is equivalent to finding the largest

utility, this is easy to compute.

3 Empirical Results

In this section, we apply our theoretical results to existing data from experiments. First, we show how our result can be applied using the Andreoni and Harbaugh (2009) task and data from Breig and Feldman (2021). We then apply the results to the "bomb" elicitation task, using data from the repeated treatment of Crosetto and Filippin (2013).

3.1 Andreoni-Harbaugh Task

We use data from Breig and Feldman (2021). In the initial part of the experiment, which is the source of the data we focus on, subjects made 50 choices from the class of budgets described in the previous section. In each of the 50 choices, subjects used a slider to allocate a budget towards either increasing the size of a prize or increasing the likelihood of receiving the prize. Manipulating the slider modified an interactive graphical feature that showed the currently selected probability and prize using a pie chart and bar chart, respectively. Subjects confirmed their choice by typing it in before moving to the next task. The maximum prizes available across budgets varied from 30 Australian dollars to 200 Australian dollars, and the maximum probabilities ranged from 0.15 to 1.

Our previous results showed that 27/181 subjects choose in a way that is consistent with maximizing a strictly increasing utility function (i.e., they have an HMI of one). Of these 27 subjects, five were consistent with expected value maximization (allocating exactly 50% of each of their budgets towards the prize), and one allocated exactly 25% of each of their budgets towards the prize. It is easy to apply our theoretical results to find the corresponding numbers of decision makers that are consistent with expected utility (7/181) and risk-averse expected utility (6/181). The only subjects who were consistent with risk-averse expected utility were those who followed the simple decision rule of allocating a constant proportion of their budget to the prize in every choice problem.

Choices from 50 budgets prove to be a demanding test for expected utility and risk-averse expected utility. A natural question is how close subjects' choices were to consistency with either model. To address this question, we compute the size of the largest subset of each subject's choices that is consistent with each model. We label these indices as EU HMI and RAEU HMI and show their distribution in Figure 2.

There are a few things to note about the distributions found in Figure 2. First, they both share the characteristic that there is a central distribution with a long tail of high scores. The scores that are equal to 1 are those subjects that are completely consistent with EU and RAEU, respectively. Because risk-averse expected utility is more restrictive than expected utility, the distribution of EU HMI (first-order) stochastically dominates RAEU HMI. However, it is stark how much lower RAEU scores are than EU. This is largely because of the restriction that for a data set to be consistent with RAEU, the subject must choose the exact same positive prize for all budgets that have the same maximum prize. Because there are ten unique maximum prizes, this requires a large degree of

set of choices consistent with risk-averse expected utility.

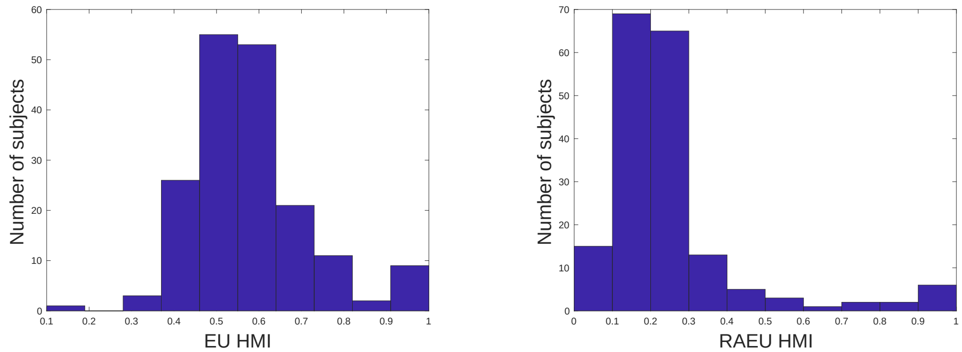

Figure 2: EU HMI and RAEU HMI in the data from Breig and Feldman (2021)

consistency across the offered budgets. Given that chosen prizes should be identical in each set of budgets sharing a common maximum prize, we observe a low median RAEU HMI of 0.22.

In Figure, 3 we compare the distributions of HMI indices from Breig and Feldman (2021) to those that arise from random choice (Bronars, 1987). Using the experiments' budgets, we create 1000 artificial data sets in which each artificial subject randomizes uniformly over the 101 available options in each budget. In general, the proportion of choices that are consistent in the data set from real subjects is much higher than the proportion from randomly generated data sets. The exception to this observation is a small number of subjects that have RAEU HMI indices much lower than those that arise from random data sets.[2]

We show the relationship between the different types of HMI score in Figure 4.[3] These measures are correlated, but not perfectly so: a subject having a higher EU HMI score is not guaranteed to have a higher RAEU HMI score.

3.2 BRET Task

Here we discuss the "bomb" risk elicitation task that was first proposed in Crosetto and Filippin (2013). The bomb elicitation task provides a visual representation for lottery choices: the subject is offered a set of boxes and can choose how many to open. One box contains a "bomb," and if opened leads to the subject earning nothing. All other boxes contain a fixed dollar amount that is added to the subject's payment (conditional on the subject not opening the box with the bomb). Many researchers have adopted this as a way to elicit risk preferences due to its intuitiveness. It is worth restating this bomb elicitation in our notation. In particular, each budget that subjects face have

[2] Of the 15 subjects with RAEU HMI scores less than or equal to 0.1, the median number of choices in which more than 50% of the budget was allocated to the prize was 38.

[3] One may also wonder the extent to which there exists a utility function that rationalizes behavior that is monotone with respect to first-order stochastic dominance (see Breig and Feldman (2021) for a discussion about the difference between utility rationalization and FOSD rationalization). In the setting described in this paper, a data set can be rationalized by such a utility function if it is rationalized by *any* strictly increasing utility function and no choices in the data set are endpoints. It is then straightforward to compute a FOSD HMI index. Because so few dominated choices are made in the data set from Breig and Feldman (2021), 160/181 subjects have a FOSD HMI index that is equal to their HMI index, and the average difference in indices is less than 0.01.

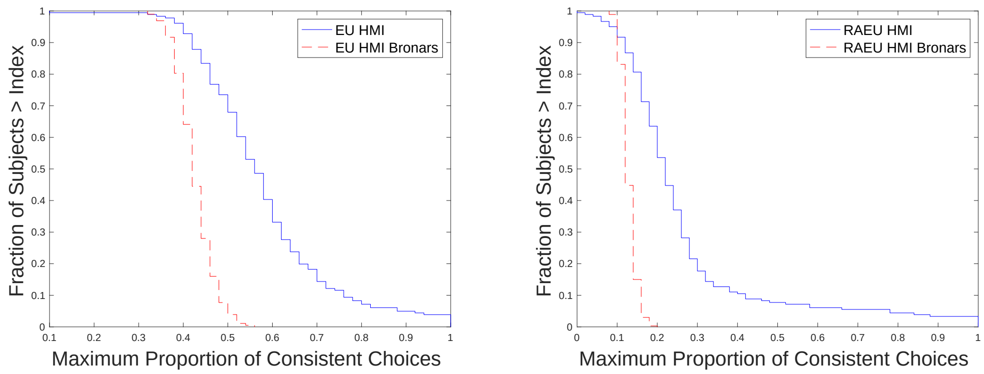

Figure 3: EU HMI and RAEU HMI in the data from Breig and Feldman (2021)

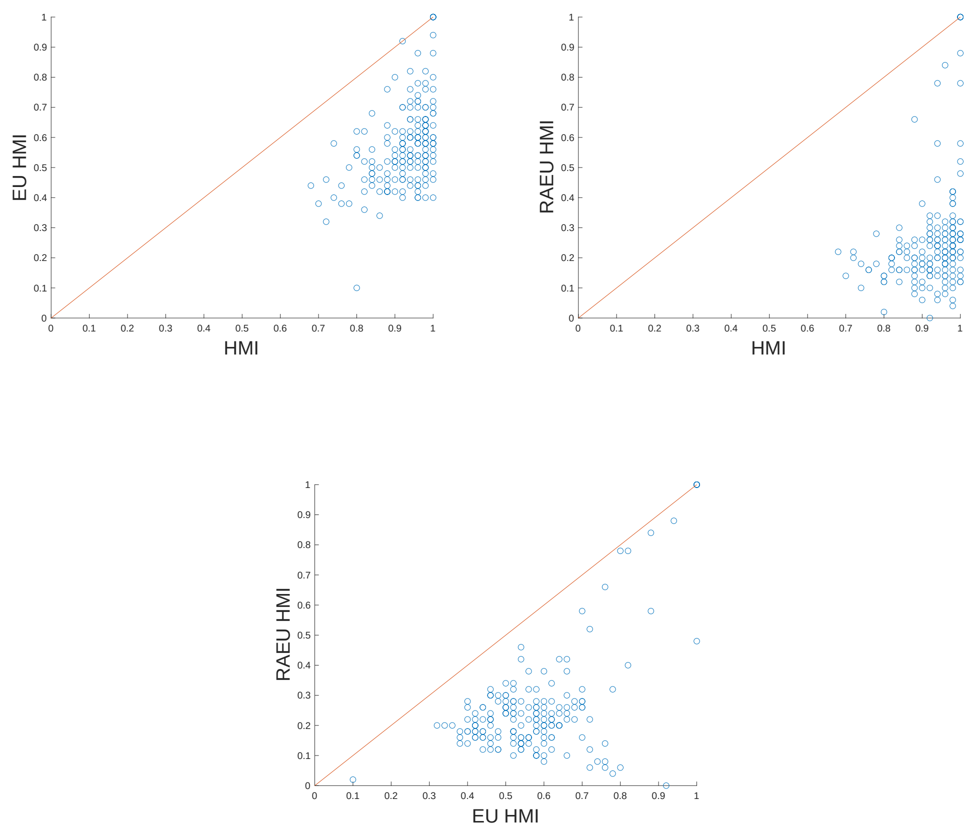

Figure 4: Relationships between HMI, EU HMI, and RAEU HMI

maximum probability of $m = 1$. The maximum prize M, on the other hand, is equal to the value of each opened box multiplied by the total number of boxes (usually 100).

While the BRET has become a popular device to measure preferences over risk, experiments rarely use within-*repetitions* of the BRET task. We use data from the "repeated" treatment used by Crosetto and Filippin (2013), in which 61 subjects faced five repetitions with the same number of boxes (100) and value per box (10 Euro cents) across repetitions. As far as we know, no studies have been completed in which subjects face multiple repetitions of the BRET task with the box value differing across treatments.

We can now apply our revealed preference results to the BRET task. As we have already mentioned, the data we use includes five repetitions of the BRET task using the same box value for each repetition. Because the maximal prize does not vary across repetitions, the *only* implication of expected utility in this experiment is that subjects do not choose the "endpoints" of their budget, which in the case of BRET involves opening either no boxes or all of the boxes. Such behavior is extremely rare in the data from Crosetto and Filippin (2013): only one subject chooses an endpoint, and does so on two of their tasks. Thus, sixty subjects have an EU HMI of 1 and one subject has an EU HMI of 0.6.[4]

While expected utility has limited implications for choices without varying maximal budgets, risk-averse expected utility places much stronger restrictions on choice data. In particular, a subject's choices are only consistent with risk-averse expected utility if they involve opening less than half of the boxes and all choices are the same. This is empirically very restrictive for the same reason risk-averse expected utility is restrictive with the Andreoni-Harbaugh task: subjects do not often make the same choice across repetitions. This is shown in Figure 5, which displays the RAEU HMI for the data from Crosetto and Filippin (2013). 35 of the 61 subjects have a RAEU HMI of either 0 or 0.2.[5] The median RAEU HMI is 0.2.[6]

4 Conclusion

We conclude with a few observations.

First, we encourage researchers that adopt the BRET to use multiple repetitions of the task with varied levels of box values. While almost all data sets are rationalizable by expected utility with a single observation, adding a single additional choice will allow for much stronger consistency tests.

Second, our theoretical and empirical results relate to a broader discussion in the revealed preference literature about the usefulness of various consistency indices (Echenique et al., 2011; Apesteguia and Ballester, 2015; Allen and Rehbeck, 2021; Echenique, 2021). Our results demonstrate that

[4] If we assume that "random choice" in the BRET tasks corresponds to opening a random number of boxes to open, it is straightforward to see that the HMI EU Bronars is equivalent to one-fifth of a random variable with binomial distribution and parameters $n = 5$ and $p = 99/101$.

[5] Again under the assumption that random choice in the BRET tasks corresponds to opening a random number of boxes, we find that over 90% of random data sets give a RAEU HMI score of 0.2, with the remaining proportion nearly equally split between scores of 0 and 0.4.

[6] One might notice that the median RAEU HMI is roughly 20% of the overall number of budgets in the data from both Breig and Feldman (2021) and Crosetto and Filippin (2013). Rather than being a deeper observation about human behavior, this is more likely due to the simple fact that the ratio number of unique maximal prizes to overall budgets is 0.2 in both studies.

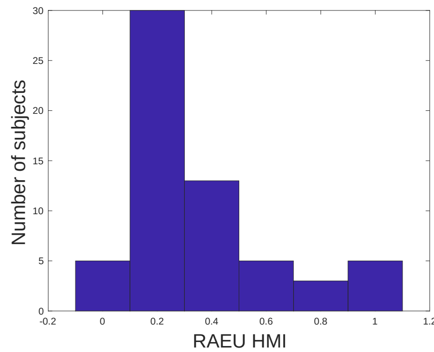

Figure 5: RAEU HMI in the data from the repeated treatment of Crosetto and Filippin (2013)

Houtman-Maks Indices may also have undesirable characteristics when applied to RAEU in linear probability-prize tradeoffs. A subject's choices under probability-prize tradeoffs can be arbitrarily close to maximizing expected value, but still have a very low RAEU HMI: a subject in the repeated treatment of Crosetto and Filippin (2013) who opened 46, 47, 48, 49, and 50 boxes would have a risk-averse expected utility HMI of only 0.2, even though the expected value maximizing choice is to open 50 boxes. Thus, some form of continuity in the rationality measure is one potential desideratum of consistency indices.

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A Proofs

A.1 Proof of Theorem 1

A.1.1 Proof of the "if" statement

The proof will be by construction. Suppose that for the regular data set $\mathcal{O}$, $0 < x_t < M_t$ for all t and $x_t \geqslant x_{t-1}$ for all $t \geqslant 2$. We define

$$u^{\mathrm{EU}}(x) = \begin{cases} \varepsilon x \text{ if } x < x_1 \\ 1 + \varepsilon x \text{ if } x_1 \leqslant x < x_2 \\ \frac{M_2 - x_1}{M_2 - x_2} + \varepsilon(x - x_2) \text{ if } x_2 \leqslant x < x_3 \\ \left(\frac{M_3 - x_2}{M_3 - x_3}\right)\left(\frac{M_2 - x_1}{M_2 - x_2}\right) + \varepsilon(x - x_3) \text{ if } x_3 \leqslant x < x_4 \\ \vdots \\ \left[\prod_{\tau=2}^{T-1} \frac{M_\tau - x_{\tau-1}}{M_\tau - x_\tau}\right] + \varepsilon(x - x_{T-1}) \text{ if } x_{T-1} \leqslant x < x_T \\ \left[\prod_{\tau=2}^{T} \frac{M_\tau - x_{\tau-1}}{M_\tau - x_\tau}\right] + \varepsilon(x - x_T) \text{ if } x_T \leqslant x \end{cases}$$

where the constant ε is taken to be an arbitrarily small but strictly positive number. Where the function u is continuous, its slope is ε. It is discontinuous for x_t such that $x_t > x_{t-1}$, with

$$\begin{aligned} \lim_{x \to x_t^-} u^{\mathrm{EU}}(x) &= u^{\mathrm{EU}}(x_{t-1}) + \varepsilon(x_t - x_{t-1}) \\ &\approx u^{\mathrm{EU}}(x_{t-1}) \\ &< \frac{m_t - x_{t-1}}{m_t - x_t} u^{\mathrm{EU}}(x_{t-1}) \\ &= u^{\mathrm{EU}}(x_t). \end{aligned}$$

The second line follows from ε being close to 0 and the third line follows from $x_t > x_{t-1}$. Thus, u^{EU} is a strictly increasing piecewise linear function with $u(0) = 0$.

We will show that for all t,

$$p_t u^{\mathrm{EU}}(x_t) \geqslant p u^{\mathrm{EU}}(x)$$

for all p and x such that $x + \frac{M_t}{m_t} p = M_t$. It will first be useful to prove a few lemmas that describe the characteristics of u^{EU}.

Lemma 1. *Suppose that conditions (1) and (2) of Theorem 1 hold and that u^{EU} is defined as above. Then for all $t < T$,*

$$(M_t - x_t) u^{EU}(x_t) \geqslant (M_t - x_{t+1}) u^{EU}(x_{t+1})$$

and for all $t > 1$.

$$(M_t - x_t)u^{EU}(x_t) \geqslant (M_t - x_{t-1})u^{EU}(x_{t-1})$$

Proof. Suppose that the first inequality stated in the lemma did not hold. Then

$$(M_t - x_t)u^{\mathrm{EU}}(x_t) < (M_t - x_{t+1})u^{\mathrm{EU}}(x_{t+1}).$$

By the definition of u^{EU}, this implies that

$$(M_t - x_t) < (M_t - x_{t+1})\left(\frac{M_{t+1} - x_t}{M_{t+1} - x_{t+1}}\right).$$

But this implies that $(M_t - x_t)(M_{t+1} - x_{t+1}) < (M_t - x_{t+1})(M_{t+1} - x_t)$, which simplifies to $(M_t - M_{t'})(x_t - x_{t'}) < 0$, a contradiction of the maintained assumptions that $\mathcal{O}$ is regular and $x_t \geqslant x_{t-1}$ for all $t \geqslant 2$.

Similarly, suppose that

$$(M_t - x_t)u^{\mathrm{EU}}(x_t) < (M_t - x_{t-1})u^{\mathrm{EU}}(x_{t-1}).$$

Then using the definition of u^{EU}, we have that

$$(M_t - x_t)\left(\frac{M_t - x_{t-1}}{M_t - x_t}\right) < (M_t - x_{t-1}),$$

but this obviously holds with equality. □

Lemma 2. *Suppose that conditions (1) and (2) of Theorem 1 hold and that u^{EU} is defined as above. Then for all t and t',*

$$(M_t - x_t)u^{EU}(x_t) \geqslant (M_t - x_{t'})u^{EU}(x_{t'}).$$

Proof. By Lemma 1, we have that

$$(M_t - x_t)u^{\mathrm{EU}}(x_t) \geqslant (M_t - x_{t+1})u^{\mathrm{EU}}(x_{t+1})$$

for all $t < T$. Thus,

$$\begin{aligned}
(M_t - x_t)u^{\mathrm{EU}}(x_t) &\geqslant (M_t - x_{t+1})u^{\mathrm{EU}}(x_{t+1}) \\
&= (M_{t+1} - x_{t+1})u^{\mathrm{EU}}(x_{t+1}) + (M_t - M_{t+1})u^{\mathrm{EU}}(x_{t+1}) \\
&\geqslant (M_{t+1} - x_{t+2})u^{\mathrm{EU}}(x_{t+2}) + (M_t - M_{t+1})u^{\mathrm{EU}}(x_{t+1}) \\
&= (M_t - x_{t+2})u^{\mathrm{EU}}(x_{t+2}) + (M_t - M_{t+1})u^{\mathrm{EU}}(x_{t+1}) + (M_{t+1} - M_t)u^{\mathrm{EU}}(x_{t+2}) \\
&= (M_t - x_{t+2})u^{\mathrm{EU}}(x_{t+2}) + (M_{t+1} - M_t)(u^{\mathrm{EU}}(x_{t+2}) - u^{\mathrm{EU}}(x_{t+1})) \\
&\geqslant (M_t - x_{t+2})u^{\mathrm{EU}}(x_{t+2}).
\end{aligned}$$

The first and second inequality follow from the fact that, as stated in Lemma 1,

$$(M_t - x_t)u^{\mathrm{EU}}(x_t) \geqslant (M_t - x_{t+1})u^{\mathrm{EU}}(x_{t+1})$$

for all $t < T$. The third inequality follows from the facts that $M_{t+1} \geqslant M_t$, $x_{t+2} \geqslant x_{t+1}$, and u^{EU} is increasing. This reasoning can be iterated to give the result for all $t' > t$.

Also by Lemma 1, we have that

$$(M_t - x_t)u^{\mathrm{EU}}(x_t) \geqslant (M_t - x_{t-1})u^{\mathrm{EU}}(x_{t-1}).$$

Similar reasoning as above gives us that

$$\begin{aligned}
(M_t - x_t)u^{\mathrm{EU}}(x_t) &\geqslant (M_t - x_{t-1})u^{\mathrm{EU}}(x_{t-1}) \\
&= (M_{t-1} - x_{t-1})u(x_{t-1}) + (M_t - M_{t-1})u(x_{t-1}) \\
&\geqslant (M_{t-1} - x_{t-2})u(x_{t-2}) + (M_t - M_{t-1})u(x_{t-1}) \\
&= (M_t - x_{t-2})u(x_{t-2}) + (M_t - M_{t-1})u(x_{t-1}) + (M_{t-1} - M_t)u(x_{t-2}) \\
&= (M_t - x_{t-2})u(x_{t-2}) + (M_t - M_{t-1})(u(x_{t-2}) - u(x_{t-2})) \\
&\geqslant (M_t - x_{t-2})u(x_{t-2}).
\end{aligned}$$

This again can be iterated, so the result holds for all $t' < t$. □

Lemma 3. *Suppose that conditions (1) and (2) of Theorem 1 hold and that u^{EU} is defined as above. Then for all t,*

$$(M_t - x_t)u^{EU}(x_t) > (M_t - x)u^{EU}(x).$$

Proof. Notice that $(M_t - x)u^{\mathrm{EU}}(x)$ is differentiable if $x \neq x_{\hat{t}}$ for any $\hat{t}$. If $x > x_1$ and we define t' as the largest value such that $x_{t'} \leqslant x$, we have that the derivative is

$$\varepsilon(M_t + x_{t'} - 2x) - u(x_{t'}),$$

which is always negative for ε small enough. Thus, in this case,

$$\begin{aligned}
(M_t - x)u^{\mathrm{EU}}(x) &< (M_t - x_{t'})u^{\mathrm{EU}}(x_{t'}) \\
&\leqslant (M_t - x_t)u^{\mathrm{EU}}(x_t),
\end{aligned}$$

where the second inequality follows from Lemma 2. If instead $x < x_1$,

$$(M_t - x)u^{\mathrm{EU}}(x) = \varepsilon(M_t x - x^2)$$

which has an upper bound that approaches zero as ε approaches 0. □

We can now complete the proof of the "if" statement. Suppose that there were a t, x, and p

such that

$$p_t u^{\text{EU}}(x_t) < p u^{\text{EU}}(x)$$

where $x + \frac{M_t}{m_t} p = M_t$. Then it must be the case that

$$\frac{m_t}{M_t}(M_t - x_t) u^{\text{EU}}(x_t) \geqslant \frac{m_t}{M_t}(M_t - x) u^{\text{EU}}(x)$$

and

$$(M_t - x_t) u^{\text{EU}}(x_t) \geqslant (M_t - x) u^{\text{EU}}(x).$$

However, this contradicts Lemma 3, so

$$p_t u(x_t) \geqslant p u(x)$$

for all p and x such that $x + \frac{M_t}{m_t} p = M_t$. Thus, $\mathcal{O}$ is rationalized by expected utility.

A.1.2 Proof of the "only if" statement

First, suppose that $x_t = 0$ or $x_t = M_t$. Then $\mathcal{O}$ cannot be rationalized by expected utility because if u is strictly increasing with $u(0) = 0$, then $p_t u(x_t) = 0$, which is strictly less than any other bundle on the interior of the budget line.

Next, suppose that $\mathcal{O}$ was rationalized by expected utility. Then for all t and t',

$$(M_t - x_t) u(x_t) \geqslant (M_t - x_{t'}) u(x_{t'})$$

and

$$(M_{t'} - x_t) u(x_t) \leqslant (M_{t'} - x_{t'}) u(x_{t'}).$$

But together, these imply that

$$(M_t - M_{t'}) u(x_t) \geqslant (M_t - M_{t'}) u(x_{t'})$$

and $(M_t - M_{t'})(u(x_t) - u(x_{t'})) \geqslant 0$. Because u is required to be strictly increasing, this implies that $x_t \geqslant x_{t'}$ if $M_t > M_{t'}$. Thus, for a regular data set, $x_t \geqslant x_{t-1}$ is a necessary condition for $\mathcal{O}$ to be rationalized by expected utility.

A.2 Proof of Theorem 2

First, notice that if a concave utility function $\hat{u}$ that rationalizes $\mathcal{O}$ exists, then there also exists a concave utility function u that rationalizes $\mathcal{O}$ where $u(x_t) = \hat{u}(x_t)$ for all t and $u(x_t)$ is piecewise linear. In this case, u gives the same utility for all of the chosen options but lower utility for any of the unchosen options. Thus, this proof will focus on whether there exists a piecewise linear utility function u that rationalizes $\mathcal{O}$ and only has kinks at the values of x that are chosen.

A.2.1 Proof of the "if" statement

The proof will be by construction. We define

$$u^{\text{RAEU}}(x) = \begin{cases} \frac{x}{x_1} \text{ if } x \leqslant x_1 \text{ if } x \leqslant x_1 \\ \frac{M_2 - 2x_2 + x}{M_2 + x_1 - 2x_2} \text{ if } x_1 < x \leqslant x_2 \\ \left(\frac{M_3 - 2x_3 + x}{M_3 + x_2 - 2x_3}\right)\left(\frac{M_2 - x_2}{M_2 + x_1 - 2x_2}\right) \text{ if } x_2 < x \leqslant x_3 \\ \vdots \\ \left(\frac{M_k - 2x_k + x}{M_k + x_{k-1} - 2x_k}\right)\left(\prod_{i=2}^{k-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right) \text{ if } x_{k-1} < x \leqslant x_k \\ \vdots \\ \left(\frac{M_T - 2x_T + x}{M_T + x_{T-1} - 2x_T}\right)\left(\prod_{i=2}^{T-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right) \text{ if } x_{T-1} < x \leqslant x_T \\ \prod_{i=2}^{T-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i} + \varepsilon(x - x_T) \text{ if } x_T < x \end{cases}$$

where the constant ε is taken to be an arbitrarily small but strictly positive number. Notice that this is a continuous piecewise linear function with $u^{\text{RAEU}}(0) = 0$. u^{RAEU} only has kinks where $x = x_t$ for some t.

First, we show that this utility function is everywhere increasing. Because both x_t and ε are positive, u^{RAEU} is strictly increasing below x_1 and above x_T. Now consider the range (x_{t-1}, x_t) for some $t > 1$. On this range, the slope of u^{RAEU} is

$$\left(\frac{1}{M_t + x_{t-1} - 2x_t}\right)\left(\prod_{i=2}^{t-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right).$$

Obviously, $M_i - x_i$ is strictly positive for all i. We can also note that if $M_t - M_{t-1} \geqslant 2(x_t - x_{t-1})$, then

$$\begin{aligned} M_t - 2x_t + x_{t-1} &\geqslant M_{t-1} - x_{t-1} \\ &> 0. \end{aligned}$$

Thus, $M_t + x_{t-1} - 2x_t$ is strictly positive for all t, and the slope must be positive.

Next we show that u^{RAEU} is concave. Because the function is piecewise linear and continuous, we only must show that the slope decreases at the kinks. Because $M_t - M_{t-1} \geqslant 2(x_t - x_{t-1})$ for all t and $x_1 \leqslant \frac{1}{2}M_1$, we have that $0 \leqslant M_1 - 2x_1 \leqslant M_2 - 2x_2$. But this implies that $M_2 + x_1 - 2x_2 \geqslant x_1$ and

$$\frac{1}{x_1} \geqslant \frac{1}{M_2 + x_1 - 2x_2}.$$

Thus, the slope decreases at x_1. Next note that the slope on the interval (x_{t-1}, x_t) is

$$\left(\frac{1}{M_t + x_{t-1} - 2x_t}\right)\left(\prod_{i=2}^{t-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right)$$

while the slope in the interval (x_t, x_{t+1}) is

$$\frac{M_t - x_t}{M_{t+1} + x_t - 2x_{t+1}}\left(\frac{1}{M_t + x_{t-1} - 2x_t}\right)\left(\prod_{i=2}^{t-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right).$$

Because $M_{t+1} - M_t \geqslant 2(x_{t+1} - x_t)$, we have that $M_{t+1} + x_t - 2x_{t+1} \geqslant M_t - x_t$, so

$$\frac{M_t - x_t}{M_{t+1} + x_t - 2x_{t+1}} \leqslant 1.$$

This implies that the slope decreases at x_t for all $t < T$. Finally, because ε is taken to be arbitrarily small, the slope decreases at x_T.

Next, we show that for all t, $p_t u^{\mathrm{RAEU}}(x_t) \geqslant p u^{\mathrm{RAEU}}(x)$ for all p and x such that $x + \frac{M_t}{m_t} p = M_t$. Note that when we substitute in the budget constraint for p, we find that the concavity of u^{RAEU} implies that

$$\frac{m_t}{M_t}(M_t - x)u^{\mathrm{RAEU}}(x)$$

is concave in x. Thus, to determine whether x_t maximizes this function, we only need to check whether it maximizes the function locally.

For $1 < t < T$, on the interval (x_{t-1}, x_t), the derivative is

$$\frac{m_t}{M_t}\left(\prod_{i=2}^{t-1} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right)\left(\frac{2(x_t - x)}{M_t + x_{t-1} - 2x_t}\right),$$

which is always positive. For $1 < t < T$, on the interval (x_t, x_{t+1}), the derivative is

$$\frac{m_t}{M_t}\left(\prod_{i=2}^{t} \frac{M_i - x_i}{M_i + x_{i-1} - 2x_i}\right)\left(\frac{M_t - M_{t+1} + 2(x_{t+1} - x)}{M_{t+1} + x_t - 2x_{t+1}}\right).$$

This is always negative, because the maintained assumption that $M_{t+1} - M_t \geqslant 2(x_{t+1} - x_t)$ implies that $0 \geqslant M_t - M_{t+1} + 2(x_{t+1} - x_t)$. Thus, for $1 < t < T$, x_t maximizes

$$\frac{m_t}{M_t}(M_t - x)u^{\mathrm{RAEU}}(x).$$

Finally, we consider the cases of $t = 1$ and $t = T$. Using the same computations as above, we can see that

$$\frac{m_T}{M_T}(M_T - x)u^{\mathrm{RAEU}}(x)$$

is increasing on the interval (x_{T-1}, x_T). For $x > x_T$, $u^{\mathrm{RAEU}\prime}(x) \approx 0$, so x_T must be the maximizer. Again using the computations from above, we can see that

$$\frac{m_1}{M_1}(M_1 - x)u^{\mathrm{RAEU}}(x)$$

is decreasing on the interval (x_1, x_2). For $x < x_1$, the derivative is

$$\frac{m_t}{M_t}\left(\frac{M_t - 2x}{x_1}\right),$$

so on that interval the derivative is always positive if $x_1 \leqslant \frac{1}{2}M_1$, which is the maintained assumption.

A.2.2 Proof of the "only if" statement

Suppose that some strictly increasing piecewise linear concave function u with kinks only at values of x_t rationalizes $\mathcal{O}$. I will show that if any of the conditions from the theorem are violated, then there is a contradiction.

First, suppose that $x_1 > \frac{1}{2}M_1$. But if u rationalizes $\mathcal{O}$, then it must be the case that

$$\frac{m_1}{M_1}(M_1 - x_1)u(x_1) \geqslant \frac{m_1}{M_1}\left(M_1 - \frac{1}{2}M_1\right)u\left(\frac{1}{2}M_1\right).$$

But because u is piecewise linear with no kinks between 0 and x_1, this implies that

$$(M_1 - x_1)u(x_1) \geqslant \frac{1}{2}M_1\left(\frac{M_1}{2x_1}\right)u(x_1)$$

and

$$(M_1 - x_1)x_1 \geqslant \frac{1}{4}M_1^2.$$

But this only holds for $x_1 = \frac{1}{2}M_1$, which is a contradiction.

Next, suppose that $x_t = 0$ for some t. If this were the case, then $p_t u(x_t) = 0$, which is strictly less than $pu(x)$ for any choice of p and x on the interior of the budget line. Thus, there is a contradiction.

The fact that $x_t - x_{t-1} \geqslant 0$ is a necessary condition for $\mathcal{O}$ to be rationalized by risk-averse expected utility follows immediately from Theorem 1.

Finally, suppose that

$$M_t - M_{t-1} < 2(x_t - x_{t-1}).$$

Because u is linear on the interval from x_{t-1} to x_t, we know that on that interval,

$$u(x) = u(x_{t-1}) + \left(\frac{x - x_{t-1}}{x_t - x_{t-1}}\right)(u(x_t) - u(x_{t-1})).$$

Thus, u rationalizing $\mathcal{O}$ implies that

$$\frac{M_{t-1}}{m_{t-1}}(M_{t-1}-x)\left[u(x_{t-1})+\left(\frac{x-x_{t-1}}{x_t-x_{t-1}}\right)(u(x_t)-u(x_{t-1}))\right]$$

is weakly decreasing at $x = x_{t-1}$, which implies that

$$u(x_t) \leqslant \frac{M_{t-1}+x_t-2x_{t-1}}{M_{t-1}-x_{t-1}}u(x_{t-1}).$$

Similarly, u rationalizing $\mathcal{O}$ must also imply that

$$\frac{M_t}{m_t}(M_t-x)\left[u(x_{t-1})+\left(\frac{x-x_{t-1}}{x_t-x_{t-1}}\right)(u(x_t)-u(x_{t-1}))\right]$$

must be weakly increasing at $x = x_t$, which implies that

$$u(x_t) \geqslant \frac{M_t-x_t}{M_t-2x_t+x_{t-1}}u(x_{t-1}).$$

We can combine these inequalities and simplify to find that

$$(M_t - M_{t-1} - 2(x_t - x_{t-1}))(x_t - x_{t-1}) \geqslant 0.$$

Thus, if $2(x_t - x_{t-1}) > M_t - M_{t-1}$, there is a contradiction.